ITEP/TH-34/10

Topological expansion of β-ensemble model and quantum algebraic geometry in the sectorwise approach

L. O. Chekhov[1], B. Eynard[2], O. Marchal[2]

[1] Steklov Mathematical Institute, ITEP and Laboratoire Poncelet, Moscow, Russia

[2] CEA, IPhT, F-91191 Gif-sur-Yvette, France, CNRS, URA 2306, F-91191 Gif-sur-Yvette, France.

Abstract

We solve the loop equations of the β-ensemble model analogously to the solution found for the Hermitian matrices $\beta = 1$. For $\beta = 1$, the solution was expressed using the algebraic spectral curve of equation $y^2 = U(x)$. For arbitrary β, the spectral curve converts into a Schrödinger equation $((\hbar\partial)^2 - U(x))\psi(x) = 0$ with $\hbar \propto (\sqrt{\beta} - 1/\sqrt{\beta})/N$. This paper is similar to the sister paper I, in particular, all the main ingredients specific for the algebraic solution of the problem remain the same, but here we present the second approach to finding a solution of loop equations using sectorwise definition of resolvents. Being technically more involved, it allows to define consistently the $\mathcal{B}$-cycle structure of the obtained quantum algebraic curve (a D-module of the form $y^2 - U(x)$, where $[y, x] = \hbar$) and to construct explicitly the correlation functions and the corresponding symplectic invariants $\mathcal{F}_h$, or the terms of the free energy, in $1/N^2$-expansion at arbitrary $\hbar$. The set of "flat" coordinates comprises the potential times t_k and the occupation numbers $\widetilde{\epsilon}_\alpha$. We define and investigate the properties of the $\mathcal{A}$- and $\mathcal{B}$-cycles, forms of 1st, 2nd and 3rd kind, and the Riemann bilinear identities. We use these identities to find explicitly the singular part of $\mathcal{F}_0$ that depends exclusively on $\widetilde{\epsilon}_\alpha$.

Contents

1 Introduction

In the contemporary mathematical physics, one can often meet the notion of quantum surfaces, which appears in many different aspects. Having no intension to describe all problems in which quantization of the very space–time coordinates takes place (which pertains mainly to string or brane models) we however stress that the main feature of most, if not all, these models is that the consideration is commonly restricted to simple geometries of sphere or torus. Observables in these theories are not the coordinates, which cease to commute with each other and satisfy some postulated quantum algebras, but objects related to representations of these algebras, because only these objects admit classical interpretation. In this paper, we propose a new approach to the description of these so-called "quantum surfaces," namely we begin with solutions of the standard one-dimensional Schrödinger equation with a polynomial potential and construct a higher genus quantum surface (which is the analogue of a classical hyperelliptic Riemann surface) for which we can define analogues of all the main notions of algebraic geometry.

This paper is an "alternative version" of our paper [5] in which the notion of the quantum algebraic geometry was introduced and which we refer to as paper I in what follows. In the both versions, the origin of quantum algebraic geometry is the same, the Schrödinger equation $((\hbar\partial)^2 - U(x))\psi(x) = 0$. The principal difference is that in this, second version, we use the sectorwise definition of all the quantities starting from the one-point resolvents, i.e., we use different solutions of the Schrödinger equation to construct these resolvents in different Stokes sectors of the complex plane. This enables us to define in a rigorous way the integrations over $\mathcal{A}$- and $\mathcal{B}$-cycles as well as to present a self-consistent procedure for constructing the correlation functions and the symplectic invariants.

The correlation functions $W_n^{(h)}(x_1, \dots, x_n)$ and the symplectic invariants F_h for any algebraic plane curve given by a polynomial equation

$$\mathcal{E}(x,y) = \sum_{i,j} \mathcal{E}_{i,j}\, x^i y^j = 0$$

were defined in [9, 12]. The invariants $\mathcal{F}_h(\mathcal{E})$ are defined in terms of algebraic geometry quantities related to the Riemann surface of equation $\mathcal{E}(x,y) = 0$. On the matrix model side, these invariants are terms of the $1/N^2$- (the genus) expansion of the free energy calculated in [4] for the one-matrix model and in [6] for the two-matrix model.

We introduce the notion of a "quantum curve" for which $\mathcal{E}(x,y)$ is a non-commutative polynomial of x and y:

$$\mathcal{E}(x,y) = \sum_{i,j} \mathcal{E}_{i,j}\, x^i\, y^j \qquad , \quad [y,x] = \hbar. \tag{1.1}$$

The notion of quantum curve is also known as *D-modules*, i.e., a quotient of the space of functions by $\operatorname{Ker}\mathcal{E}(x,y)$, where $y = \hbar\partial/\partial x$.

Our construction is based on the functions $\psi(x)$ such that

$$\mathcal{E}(x, \hbar\partial_x)\cdot\psi(x) = 0 \tag{1.2}$$

and we show that one can consistently define all the basic notion of algebraic geometry within this construction. Whereas some objects, like branch points, become obsolete, we can define cycles, forms, Bergman kernel, period matrix and the corresponding Abel maps as well as other objects in a consistent way.

But, otherwise, it is striking to find that almost all relationships of classical algebraic geometry remain unchanged when $\hbar \neq 0$, for instance, the Riemann bilinear identity, the modified Rauch variational formula, and the topological recursion defining the correlation functions and the symplectic invariants.

The symplectic invariants $\mathcal{F}_h$ were first introduced for the solution of loop equations arising in the 1-hermitian random matrix model [9, 4]. They were later generalized to other hermitian multi–matrix models [6, 13].

The models that correspond to the quantum surface are the β-*ensembles* classified by the exponent β. The three Wigner ensembles (see [17], and we changed $\beta \to \beta/2$) correspond to $\beta = 1$ (hermitian case), $\beta = 1/2$ (real symmetric case), $\beta = 2$ (real self-dual quaternion case), but we can easily define a β-ensemble eigenvalue model for any real value of β as the N-fold integral of the form

$$\int d\lambda_1 \cdots d\lambda_N |\Delta(\lambda)|^{2\beta} e^{-N\sqrt{\beta}\sum_{j=1}^{N} V(\lambda_j)}$$

(Δ is the Vandermonde determinant).

In [3], the solution of [9] was generalized to the β-ensembles, but the solution was presented as a double half-infinite sum for $\beta = O(1)$ at large N,

$$\mathcal{F} = \sum_{h,k=0}^{\infty} N^{2-2h-k} \, (\sqrt{\beta} - 1/\sqrt{\beta})^k \mathcal{F}_{h,k}. \tag{1.3}$$

The coefficients $\mathcal{F}_{h,k}$ were computed in [3].

In this paper, as in paper I, we assume that $\hbar = (\sqrt{\beta} - 1/\sqrt{\beta})/N$, so we perform an (infinite) resummation in the above formula; the free-energy expansion then acquires the standard form,

$$\mathcal{F} = \sum_{h=0}^{\infty} N^{2-2h} \mathcal{F}_h(\hbar). \tag{1.4}$$

The $\mathcal{F}_{h,k}$'s of [3] can be recovered by computing the semi-classical small $\hbar$-expansion of $\mathcal{F}_h(\hbar)$. We demonstrate that $\mathcal{F}_h(\hbar)$ is the natural generalization of the symplectic invariants of [12] for a "quantum spectral curve" $\mathcal{E}(x, y)$ with $[y, x] = \hbar$.

We define also analogues of the multi-point resolvents

$$W_n(x_1, \ldots, x_n) = N^{-n} \left\langle \sum_{j=1}^{N} (x_1 - \lambda_j)^{-1} \cdots \sum_{j=1}^{N} (x_n - \lambda_j)^{-1} \right\rangle_{conn},$$

where we let angular brackets denote the averaging with the weight $|\Delta(\lambda)|^{2\beta} e^{-N\sqrt{\beta}\sum_{j=1}^{N} V(\lambda_j)}$. These resolvents in turn admit the $1/N^2$-expansion, $W_n(x_1, \ldots, x_n) = \sum_{h=0}^{\infty} N^{2-2h-n} W_n^{(h)}(x_1, \ldots, x_n)$, and we calculate all the terms $W_n^{(h)}$ using the modified diagrammatic technique.

The main tool applied for studying the β-eigenvalue model is the loop equation method. We obtain loop equations from the invariance of an integral under the special change of variables. Loop equations for the β-eigenvalue model can be found in [7], [10], and here we solve them order by order in $1/N^2$, at fixed $\hbar$.

Recently, models of this type got a new vim due to the conjecture by Alday, Gaiotto, and Tachikawa (AGT) [1] relating Nekrasov's instanton function [18] to conformal blocks of the Liouville theory; these conformal blocks in turn can be described by the matrix-like model (see [16], [8]); the relation to the Nekrasov's $\epsilon_{1,2}$ parameters is explicit: $\epsilon_1\epsilon_2 \sim 1/N^2$ and $\epsilon_1/\epsilon_2 \sim \beta$, so using the approach in this paper, we can construct *nonperturbative* solutions of Nekrasov's formulas in ϵ_1/ϵ_2. In this paper, we investigate only the case of polynomial potentials, the generalization to the realistic logarithmic potentials appearing in the AGT conjecture will follows.

The structure of the paper is as follows: we collect the generalities on the Stokes phenomenon pertaining to solutions of the Schrödinger equation in Sec. 2. We describe our quantum Riemann surface in Sec. 3 where we introduce $\mathcal{A}$- and $\mathcal{B}$-cycles, filling fractions $\widetilde{\epsilon}_i$, and the first-kind functions (analogues of holomorphic and Krichever–Whitham meromorphic differentials) as well as the system of flat coordinates and the Riemann period matrix. In Sec. 4, we introduce the recursion kernels and the second- and third-kind (bi-)differentials. In Sec. 5, we go beyond the leading approximation in $1/N^2$ and construct correlation functions of all orders using the Feynman-like diagrammatic technique. We reveal the origin of our recursion procedure in Sec. 6, where we develop in details the variations w.r.t. the set of flat coordinates; the summary is in Sec. 7. In the next two (completely new as compared to paper I) sections, we investigate the link to the β-ensemble models (Sec. 8) and construct on the base of this analysis the free-energy terms (Sec. 9). In the first three appendixes to the paper, we present proofs of the three main theorems of Sec. 5 concerning properties of the correlation functions whereas the fourth appendix contains the new formula expressing $\mathcal{F}_0$ through the filling fractions $\widetilde{\epsilon}_i$; in the matrix model approach, the singular term has the structure $\frac{1}{2}\widetilde{\epsilon}_i^2 \log \widetilde{\epsilon}_i$ whereas in the quantum geometry this term is proportional to $\int \log \Gamma(\widetilde{\epsilon}_i)$, which is the first actual example of calculations in the case of quantum Riemann surfaces.

2 Schrödinger equation and resolvents

2.1 Solutions of the Schrödinger equation

We begin with the Schrödinger equation

$$\hbar^2 \psi''(x) = U(x)\,\psi(x) \tag{2.1}$$

with $U(x)$ being a polynomial of even degree $2d$ for which we define the polynomial "potential" $V(x)$ of degree $d+1$ to be

$$V'(x) = 2\,(\sqrt{U})_+ = \sum_{k=0}^{d} t_{k+1}\, x^k \tag{2.2}$$

We also define the polynomial of degree $d-1$,

$$P(x) = \frac{V'^2(x)}{4} - U(x) - \hbar \frac{V''(x)}{2}. \tag{2.3}$$

Eventually, we define:

$$t_0 = \lim_{x\to\infty} \frac{xP(x)}{V'(x)} \tag{2.4}$$

In the matrix model language (see section 8), $t_1, \ldots, t_{d+1}$ are called the *times* associated to the potential $V(x)$, t_0 is the normalized total number of eigenvalues (particles), or the temperature, whereas the remaining coefficients of P are defined by introducing fixed "filling fractions" ϵ_i below.

2.1.1 Stokes Sectors

A function $\psi(x)$ that is a solution of the Schrödinger equation exhibits the Stokes phenomenon, i.e., although $\psi(x)$ is an entire function, its asymptotics are discontinuous near ∞ where it has an essential singularity. Let $\theta_0 = \mathrm{Arg}(t_{d+1})$ be the argument of the leading coefficient of the potential $V(x)$. We define the Stokes half-lines the asymptotic directions along which $\mathrm{Re}V(x)$ vanishes asymptotically, $L_k = \left\{ x \ / \ \mathrm{Arg}(x) = -\frac{\theta_0}{d+1} + \pi \, \frac{k+\frac{1}{2}}{d+1} \right\}$, together with the corresponding Stokes sectors:

$$S_k = \left\{ \mathrm{Arg}(x) \in \left] -\frac{\theta_0}{d+1} + \pi \, \frac{k-\frac{1}{2}}{d+1}, -\frac{\theta_0}{d+1} + \pi \, \frac{k+\frac{1}{2}}{d+1} \right[\right\} \tag{2.5}$$

i.e., S_k is the sector between L_{k-1} and L_k.

Notice that in even sectors we have asymptotically $\mathrm{Re}V(x) > 0$ and in odd sectors we have $\mathrm{Re}V(x) < 0$.

2.1.2 The Stokes phenomenon. Decreasing solution

From the study of the Schrödinger equation it is known that $\psi(x)$ is an entire function having a large x expansion in each sector S_k,

$$\psi(x) \underset{S_k}{\sim} e^{\pm \frac{1}{2\hbar} V(x)} \, x^{C_k} \, \left(A_k + \frac{B_k}{x} + \ldots\right) \tag{2.6}$$

and the sign $\pm$, may jump discontinuously from one sector to another as well as the numbers $A_k, B_k, C_k, \ldots$ (and in general, all the coefficients of the series in $\frac{1}{x^j}$ at infinity).[1]

In every sector S_k there exists a unique solution that decreases exponentially along each direction inside the sector. We now separate solutions in the even and odd sectors

[1] The corresponding series is asymptotic, so we cannot continue it analytically to other sectors.

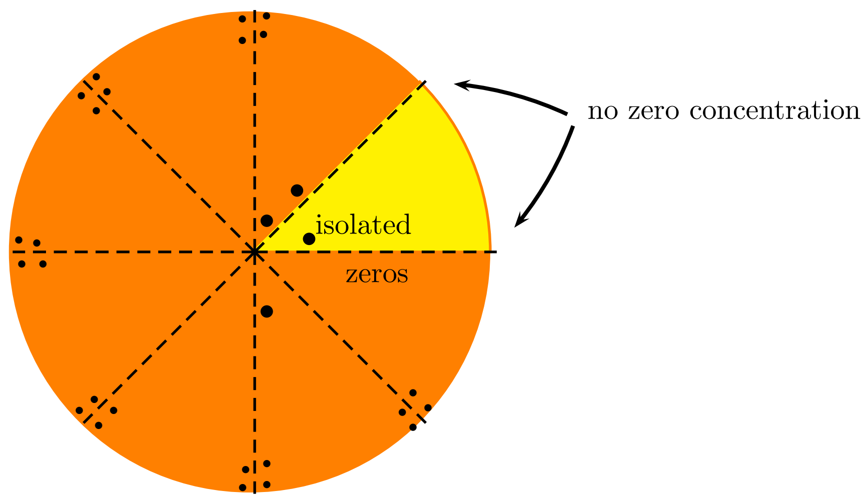


Figure 1: Example of the Stokes sector partition and structure of zeros for the Schrödinger equation solution $\psi(x)$ that decreases in the light-colored sector and increases in all other sectors (the degree of the potential $V(x)$ is four).

and consider the set $\{\psi_\alpha(x)\}$ of solutions each of which decreases in the corresponding even sector. We therefore introduce a *sectorwise* system of solutions to the Schrödinger equation.

An important and useful result is the Stokes theorem, which claims that if the asymptotics of $\psi(x)$ is exponentially small in some sector, then the same asymptotic series expansion (2.6) is valid in the two adjacent sectors (and therefore $\psi(x)$ is exponentially large in those two sectors).

In the general case, (i.e., for a generic potential $U(x)$), the solution $\psi_\alpha(x)$ decreases only in the sector S_α, and is exponentially large in all other sectors. But if the Schrödinger potential $U(x)$ is non-generic, then there may exist several sectors in which $\psi_\alpha(x)$ is exponentially small (which means that $\psi_{\alpha_1}(x) = \psi_{\alpha_2}(x)$ for some $\alpha_1 \neq \alpha_2$).

In what follows, we mainly consider the general case, so in what follows we assume all the functions ψ_α to be different if not stating the opposite.

The case studied in [11] was the most degenerate case in which one and the same solution ψ is exponentially small in $d+1$ sectors.

2.1.3 Zeroes of ψ

Every $\psi_\alpha(x)$ is an entire function with an essential singularity at ∞, and with isolated zeroes $s_i^{(\alpha)}$, $\psi_\alpha(s_i^{(\alpha)}) = 0$. The number of these zeros can be finite or infinite. In the latter case, zeroes may only accumulate near ∞, and only along the Stokes half-lines L_j bordering the sectors (see fig.2.1.1). This accumulation of zeroes along the half–line L_j occurs if and only if $\psi_\alpha(x)$ is exponentially large on both sides of the half-line. So, no accumulation of zeros of the function $\psi_\alpha(x)$ occurs along the lines that border the α's sector, and this function can therefore have only a finite number of zeros inside the "bigger" sector when we join the α's sector with the adjacent parts of the two neighboring sectors.

If $U(x)$ is generic, then each of $\psi_\alpha(x)$ has an infinite number of zeroes, the zeroes accumulate at ∞ along all half-lines L_j with $j \neq \alpha, \alpha-1$.

In paper I, we define the genus of the Schrödinger equation to be related to the number of half-lines of zeros accumulation of a selected function ψ_0. However, this definition is scheme-dependent, and we can in principle obtain different genera for the very same function $U(x)$. The clear understanding of this is still lacking; a possible explanation is that we actually deal with different sections of an ambient infinite-genus quantum surface.

2.1.4 Sheets

In sector S_α we have the asymptotic behavior

$$\psi_\alpha(x) \sim e^{-\hbar V(x)/2}\, x^{t_0/\hbar}(A_\alpha + \frac{B_\alpha}{x} + \ldots), \tag{2.7}$$

and the function ψ_α has the same asymptotic behavior in the two adjacent sectors.

We consider an α's sheet of the quantum Riemann surface to be the union of these three sectors with possible analytic continuation into a finite domain of the complex plane. We consider only the sheets enumerated by even α and, in contrast to paper I, introduce democracy of sheets: all of them will be equivalent in the approach of this paper. Sheets obviously overlap; we have to choose boundaries (cuts) between them.

2.2 Resolvent

The first ingredient of our strategy is to define a resolvent similar to the one in matrix models.

Definition 2.1 *We define the resolvent sectorwise:*

$$\omega(\overset{\alpha}{x}) = \hbar\frac{\psi_\alpha'(x)}{\psi_\alpha(x)} + \frac{V'(x)}{2},\ \textit{for } x \in S_\alpha. \tag{2.8}$$

For a quantity defined sectorwise we indicate it by setting the sector index above the variable, as shown in (2.8).

It follows from this definition that $\omega(x)$ has simple poles at zeros of ψ_α in the corresponding sector. The boundaries between sectors overlap, but in what follows we fix them in a more explicit form (see the partition of the complex plane by $\mathcal{A}$-cycles).

A straightforward computation then gives

$$\omega(\overset{\alpha}{x}) \underset{x\to\infty_\alpha,\infty_{\alpha\pm1}}{\sim} \frac{t_0}{x} + O(1/x^2), \tag{2.9}$$

that is, in each sheet the resolvent possesses asymptotic properties of a standard matrix-model resolvent.

An important property of any solution ψ_α is that

$$\operatorname*{Res}_{s_i^{(\alpha)}} \frac{1}{\psi_\alpha^2(x)} = 0 \tag{2.10}$$

The main property of $\omega(\overset{\alpha}{x})$ is that it satisfies the Ricatti equation. We obtain

$$\begin{aligned} V'(x)\omega(\overset{\alpha}{x}) - \omega^2(\overset{\alpha}{x}) - \hbar\omega'(\overset{\alpha}{x}) &= \frac{V'(x)^2}{4} - \hbar^2\frac{\psi''_\alpha(x)}{\psi_\alpha(x)} - \hbar\frac{V''(x)}{2} \\ &= \frac{V'(x)^2}{4} - U(x) - \hbar\frac{V''(x)}{2} \\ &= P(x), \end{aligned} \tag{2.11}$$

with $P(x)$ being a polynomial of degree $d-1$ in x and this polynomial is one and the same for all sheets of the quantum Riemann surface introduced below.

3 Quantum Riemann Surface

In this section we define the notions of $\mathcal{A}$- and $\mathcal{B}$-cycles and the first kind differentials dual to them.

3.1 The contour $\mathcal{C}_D$ and the set of $\mathcal{A}$- and $\mathcal{B}$-cycles

In papers on matrix models (on an early stage, before coming to residues at the branch points), we have the special contour of integration, $\mathcal{C}_D$, that encircles all the singularities of resolvents leaving apart all other possible singular points. The analogue of such a contour in our case is the union of $d+1$ contours, one per each sheet, that pairwise coincide in far asymptotic domains of odd Stokes sectors and separate all the zeros of the function ψ_α from the infinity ∞_α (which is always possible because we have a finite number of zeros in each sheet). We have (see Fig. 3.1)

$$\oint_{\mathcal{C}_D} f(x)dx \equiv \sum_\alpha \int_{\infty_{\alpha-1}}^{\infty_{\alpha+1}} f(\overset{\alpha}{x})dx \tag{3.1}$$

for *any* function $f(\overset{\alpha}{x})$ that has no asymptotic zero concentration along the boundary lines of the sector S_α. Here and hereafter, we assume that $f(\overset{\alpha}{x})$ may depend on a finite number of derivatives of the function $\psi_\alpha(x)$, the symbol $f(\overset{\alpha}{x})$ then indicates that we substitute the solution $\psi_\alpha(x)$ as an argument.

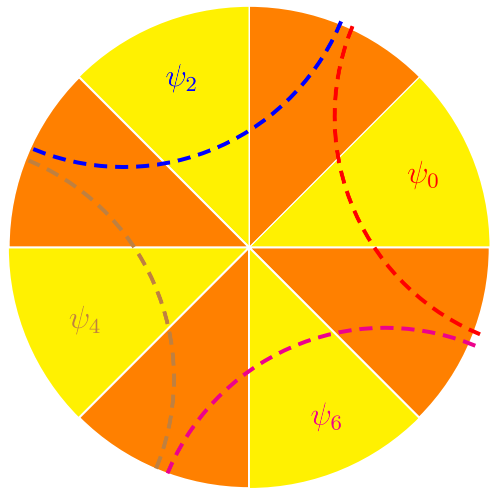


Figure 2: The original integration contour $\mathcal{C}_D$.

3.1.1 $\mathcal{A}$- and $\mathcal{B}$-cycles

We now deform the integration contour $\mathcal{C}_D$ pushing through the "middle" part of the complex plane and taking the residues at the zeros $s_i^{(\alpha)}$ of the corresponding functions ψ_α as shown in Fig. 3.1.1. On the way we might break some contours presenting them as the unions of newly introduced contours all of which are stretched between different asymptotic directions. As a result, we obtain a system of exactly $2d$ contours in which (leaving aside the residues at zeros $s_i^{(\alpha)}$) all the contours are pairwise identified and represent edges of d "cuts". As the result, we obtain a complete system of d cuts $\widetilde{\mathcal{A}}_i$, $i = 1, \dots, d$, that separate all the odd-numbered infinities[2] and determining the corresponding sheets of the quantum Riemann surface. If the functions $\psi_\alpha(x)$ coincide for some sheets, then we can identify these sheets. Note that we definitely have an arbitrariness in constructing this system of cuts; we can also arbitrarily assign the residues inside the sheet to belong to one of several contours bounding this sheet.

We call the cut separating two sheets a *cycle* $\tilde{\mathcal{A}}_\alpha$, and it is characterized by four indices: α_+ and α_- are indices of the sheets separated by this cut (they are even numbered in our classification); $\widetilde{\alpha}_+$ and $\widetilde{\alpha}_-$ are indices of infinities that are asymptotic for this cut (they are odd numbered).

To each complete set $\{\widetilde{\mathcal{A}}_\alpha\}_{\alpha=1}^d$ of $\widetilde{\mathcal{A}}$-cycles we uniquely set into the correspondence the set $\{\widetilde{\mathcal{B}}_\alpha\}_{\alpha=1}^d$ of $\widetilde{\mathcal{B}}$-cycles that go pairwise between the even-numbered infinities (∞_{α_+}

[2]In what follows, we identify an infinity "point" with the corresponding number with the related asymptotic direction.

and ∞_{α_-}) such that the intersection index $\widetilde{\mathcal{A}}_\alpha \circ \widetilde{\mathcal{B}}_\beta = \delta_{\alpha,\beta}$.

Definition 3.1 *We define the integrals over the cycles $\tilde{\mathcal{A}}_\alpha$ and the conjugate cycle $\tilde{\mathcal{B}}_\alpha$ to be (see Fig. 3.1.1)*

$$\oint_{\tilde{\mathcal{A}}_\alpha} f(x)dx \stackrel{\text{def}}{=} \int_{\infty_{\tilde{\alpha}_-}}^{\infty_{\tilde{\alpha}_+}} (f(\overset{\alpha_+}{x}) - f(\overset{\alpha_-}{x}))\, dx + \sum \underset{s_i^{(\alpha\pm)}(\alpha)}{res} f(\overset{\alpha_\pm}{x}) \qquad (3.2)$$

and

$$\oint_{\tilde{\mathcal{B}}_\alpha} f(x)dx \stackrel{\text{def}}{=} \int_{\infty_{\alpha_-}}^{\infty_{\alpha_+}} (f(\overset{\alpha_+}{x}) - f(\overset{\alpha_-}{x}))\, dx, \qquad (3.3)$$

where the residues in the first expression are taken at those zeros of $\psi_{\alpha_\pm}$ that are assigned to the corresponding contour.

Because the prescription for the sheet assignment follows from the definitions (3.2) and (3.3) of the cycle integrals, we omit the sheet labels in the corresponding integrands.

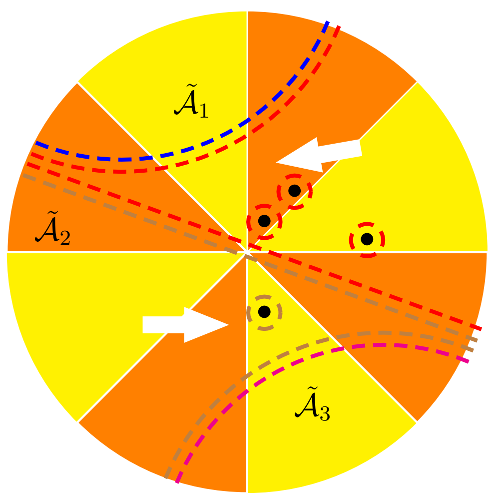


Figure 3: Example of pushing the contour $\mathcal{C}_D$ from infinities to the set of $\tilde{\mathcal{A}}$-cycles.

Remark 3.1 Assignment of residues in the αth sheet to the contours bounding this sheet is arbitrary; we have therefore a (discrete) ambiguity in the definition (3.2) of the $\tilde{\mathcal{A}}$-cycle integrals. However, the notion of the integral over $\mathcal{C}_D$ is well defined and does not depend on the choice of $\tilde{\mathcal{A}}$-cycles.

Obviously, $\oint_{\mathcal{C}_D} f(x)dx = \sum_{i=1}^d \oint_{\tilde{\mathcal{A}}_i} f(x)dx$.

We now introduce the "genuine" $\mathcal{A}$- and $\mathcal{B}$-cycles, which are straightforward analogues of the set of $\mathcal{A}$- and $\mathcal{B}$-cycles on a standard Riemann surface. For this, we select one among the $\widetilde{\mathcal{A}}$-cycles, say, the cycle $\widetilde{\mathcal{A}}_d$ and the conjugate cycle $\widetilde{\mathcal{B}}_d$. Then, we identify $\mathcal{A}_i = \widetilde{\mathcal{A}}_i$ and $\mathcal{B}_i = \widetilde{\mathcal{B}}_i - \widetilde{\mathcal{B}}_d$ for $i = 1, \ldots, d-1$ in the sense of Definition 3.1, that is

$$\begin{aligned} &\oint_{\mathcal{A}_i} f(x)dx \stackrel{\mathrm{def}}{=} \oint_{\tilde{\mathcal{A}}_i} f(x)dx, \\ &\oint_{\mathcal{B}_i} f(x)dx \stackrel{\mathrm{def}}{=} \oint_{\tilde{\mathcal{B}}_i} f(x)dx - \oint_{\tilde{\mathcal{B}}_d} f(x)dx \quad \text{for} \quad i = 1, \ldots, d-1, \end{aligned} \tag{3.4}$$

and we call the number $d - 1 = g$ of independent $\mathcal{A}$- and $\mathcal{B}$- cycles the *genus* of the quantum Riemann surface.

The newly introduced $\mathcal{A}$- and $\mathcal{B}$-cycles again satisfy the standard intersection formula,

$$\mathcal{A}_\alpha \cap \mathcal{B}_\beta = \delta_{\alpha,\beta}, \tag{3.5}$$

and most of our construction features depend only on the homology class of the paths $\mathcal{A}_\alpha$, $\mathcal{B}_\alpha$ at the asymptotic infinities, but in the intermediate considerations it is useful to choose a representant, the intersection point P_α,

$$\mathcal{A}_\alpha \cap \mathcal{B}_\alpha = \{P_\alpha\}. \tag{3.6}$$

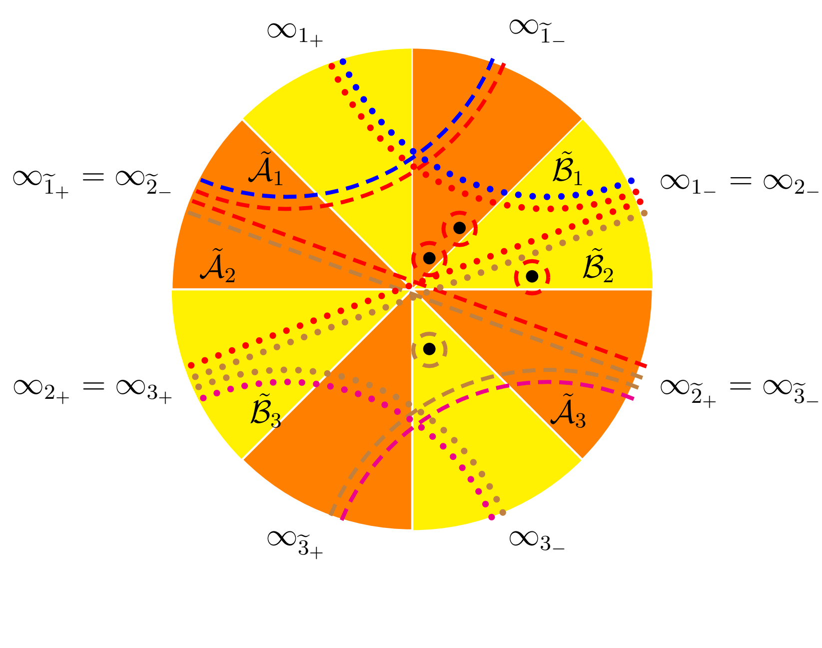


Figure 4: The pattern of $\tilde{\mathcal{A}}$- (dashed lines) and $\tilde{\mathcal{B}}$- (dotted lines) cycles for the example in Fig. 3.1.1.

3.2 Filling fractions

In random matrices, the notion of filling fractions is just the $\mathcal{A}$-cycle integrals of the resolvent. Then, if the $\mathcal{A}$-cycles are chosen to lie in the physical sheet (which is possible, say, in the hyperelliptic case), the discontinuity of the resolvent along the corresponding cuts determines the eigenvalue density and the $\mathcal{A}$-cycle integrals determine the portions of eigenvalues lying on the corresponding interval of eigenvalue distribution. They are called therefore the filling fractions.

In the case of quantum surface, we define the "filling fractions" $\widetilde{\epsilon}_\alpha$ to be

$$\widetilde{\epsilon}_\alpha = \frac{1}{2i\pi}\oint_{\widetilde{\mathcal{A}}_\alpha}\omega(x)dx \stackrel{\text{def}}{=} \int_{\infty_{\widetilde{\alpha}_-}}^{\infty_{\widetilde{\alpha}_+}}(\omega(\overset{\alpha_+}{x}) - \omega(\overset{\alpha_-}{x}))dx, \quad \alpha = 1,\dots,d. \tag{3.7}$$

Note that this definition depends on where we place the contours and (in the case where a sheet is bounded by more than one $\widetilde{\mathcal{A}}$-cycle) we also have a freedom to assign residues inside the sheet to different $\widetilde{\mathcal{A}}$-cycles, so the filling fractions are defined up to integers times $\hbar$.

For the difference, we have

$$\omega(\overset{\alpha_+}{x}) - \omega(\overset{\alpha_-}{x}) = \frac{w_{\alpha_+,\alpha_+}}{\psi_{\alpha_+}(x)\psi_{\alpha_-}(x)},$$

where $w_{\alpha_+,\alpha_+} = \psi'_{\alpha_+}\psi_{\alpha_-} - \psi'_{\alpha_-}\psi_{\alpha_+}$ is the Wronskian of the two solutions. Therefore, this difference decreases exponentially in sectors where the both solutions ψ_{α_+} and ψ_{α_-} increase, and this is why we identify the asymptotic domains of the $\mathcal{A}$-cycle integrals with "branch points."

We have

$$\sum_{\alpha=1}^{d}\widetilde{\epsilon}_\alpha = t_0, \tag{3.8}$$

which just follows from that the sum of integrals over $\widetilde{\mathcal{A}}$-cycles is equivalent to evaluating the integral over $\mathcal{C}_D$. This also means that we should take as independent variables only $d-1=g$ of the variables $\widetilde{\epsilon}_\alpha$ if we consider t_0 to be an independent variable, and we naturally choose these g variables ϵ_α to be filling fractions corresponding to the cycles $\mathcal{A}_\alpha$ of the quantum Riemann surface.

Remark 3.2 In the case $g=-1$ in [5], the only filling fraction is $\widetilde{\epsilon}_d = t_0$, and it is given by the (finite) sum of residues of the function ω at the zeros s_i:

$$\widetilde{\epsilon}_d = t_0 = \sum_i \operatorname*{Res}_{s_i}\omega = \hbar\,\#\{s_i\},$$

so t_0 is discrete in this case. For $g \geq 0$, the variables $\widetilde{\epsilon}_\alpha$, $\alpha = 1,\dots,g$, and t_0 may take arbitrary, not necessarily integer, values.

3.3 First kind functions

After defining the cycles, another important step is to define the equivalent of the first, second and third kind differentials. We begin with the definition of the first-kind differentials.

Let h_k, $k = 1, \ldots, d-1$, be a basis in the complex vector space of polynomials of degree $\leq d-2$.

We introduce the functions

$$v_k(\overset{\alpha}{x}) = \frac{1}{\hbar\, \psi_\alpha^2(x)} \int_{\infty_\alpha}^{x} h_k(x')\, \psi_\alpha^2(x')\, dx'. \tag{3.9}$$

We use the same polynomial $h_k(x')$ for all the sheets of the Riemann surface.

Note that because every $\psi_\alpha(x)$ is a solution to the Schrödinger equation, $v_k(\overset{\alpha}{x})$ has double poles with no residue at the $s_j^{(\alpha)}$ (at the zeroes of ψ_α), and behaves like $O(1/x^2)$ in the sector S_α and inside all the sectors in which ψ_α is exponentially large (if the polynomial $h_k(x')$ has power less than $d-2$). Therefore, the following integrals are well defined in the general case:

$$I_{k,\alpha} = \oint_{\mathcal{A}_\alpha} v_k(x)\, dx \quad \alpha = 1, \ldots, g,\, k = 1, \ldots, d-1. \tag{3.10}$$

If the matrix $I_{k,\alpha}$ with $k, \alpha = 1, \ldots, d-1$ has the full rank (which we assume in what follows), then it is possible to choose the canonically normalized basis of h_k such that

$$I_{k,\alpha} = \delta_{k,\alpha}. \tag{3.11}$$

The functions $v_k(x)\, k = 1, \ldots, g$ are therefore the natural analogues of canonically normalized holomorphic forms (1st kind differentials).

We now extend this notion to the *meromorphic* (Whitham–Krichever) [15] differentials. For this, let us consider the following basis h_k, $k = 1, \ldots$, in the space of polynomials of arbitrary order: the first $d-1$ elements of this basis are the original polynomials h_k each of which has degree not higher than $d-2$, each polynomial h_k with $k > d-1$ has degree exactly $k-1$ and must be chosen on the following grounds.

Define the functions $v_k(\overset{\alpha}{x})$ with $k > d-1$ exactly as in (3.9) with h_k being now a polynomial of arbitrary (but fixed) degree $k-1$. The coefficients of h_k with $k \geq d-1$ are unambiguously fixed by the normalization conditions:

- (the residue condition)

$$\oint_{\mathcal{C}_D} x^{-l}\, v_k(x) dx = \delta_{l,k-d}, \quad l = 0, 1, \ldots,\; k \geq d-1. \tag{3.12}$$

- (the normalizing condition)

$$\oint_{\mathcal{A}_\alpha} v_k(x)dx = 0,\ \alpha = 1, \ldots, d-1,\ k = d, \ldots. \tag{3.13}$$

Remark 3.3 Although the functions $v_k(\overset{\alpha}{x})$ generally increase as x^{k-d} as $x \to \infty$, the integral (3.12) as well as the normalizing condition (3.13) are well defined for any finite l and k. This is because the difference $v_k(\overset{\alpha_+}{x}) - v_k(\overset{\alpha_-}{x})$ is exponentially small as $x \to \infty_{\tilde{\alpha}_\pm}$ for any k, and we can integrate it along $\mathcal{C}_D$ weighted by any polynomially growing function. The integral (3.12) is therefore a natural analogue of the residue at infinity of order $l+1$.

3.4 Riemann matrix of periods

An interesting quantity in standard algebraic geometry is the Riemann matrix of periods provided by integrals of the holomorphic differentials over $\mathcal{B}$-cycles. So, an analogous "quantum" Riemann period matrix $\tau_{i,j}$, $i, j = 1, \ldots, g$ is

$$\tau_{\alpha,i} \stackrel{\text{def}}{=} \oint_{\mathcal{B}_\alpha} v_i(x)dx. \tag{3.14}$$

Note that this definition makes sense since $v_i(x)$ $(i = 1, \ldots, g)$ behaves as $O(1/x^2)$ in the sectors asymptotic for the $\mathcal{B}$-cycles. And because the residues of $v_i(\overset{\alpha}{x})$ vanish at all zeros $s_j^{(\alpha)}$, these integrals depend only on the homology class of $\mathcal{B}$-cycles.

Like for the classical Riemann matrix of periods we have the following property:

Theorem 3.1 *The period matrix τ is symmetric: $\tau_{i,j} = \tau_{j,i}$.*

proof:

This result follows from Theorem 4.8 below, since:

$$\oint_{\mathcal{B}_\beta} dx \oint_{\mathcal{B}_\alpha} B(x,z)dz = 2i\pi \oint_{\mathcal{B}_\beta} dx v_\alpha(x) = 2i\pi\tau_{\beta,\alpha}$$

and from the symmetry theorem 4.9 for the Bergman kernel, $B(\overset{\alpha}{x}, \overset{\beta}{z}) = B(\overset{\beta}{z}, \overset{\alpha}{x})$. $\square$

4 Recursion kernels

One of the key geometric objects in [11] and in [12], is the "recursion kernel" $K(x,z)$. It was used in the context of matrix models [6] for constructing a solution of loop equations. We use its analogue below for constructing the 3rd and 2nd kind differentials.

4.1 The recursion kernel K

First we define the kernel

$$\widehat{K}(\overset{\alpha}{x}, z) = \frac{1}{\hbar\psi_\alpha^2(x)} \int_{\infty_\alpha}^{x} \psi_\alpha^2(x')\, \frac{dx'}{x'-z} \tag{4.1}$$

and for each $\alpha = 1, \dots, g$, we define

$$\hbar C_\alpha(z) = \oint_{\mathcal{A}_\alpha} \widehat{K}(x, z) \equiv \int_{\infty_{\tilde{\alpha}_-}}^{\infty_{\tilde{\alpha}_+}} (K(\overset{\alpha_+}{x}, z) - K(\overset{\alpha_-}{x}, z)). \tag{4.2}$$

In these expressions, we must also specify the contours of integration w.r.t. the variable x' from the infinities $\infty_{\alpha_\pm}$ to the point x lying on the cycle $\mathcal{A}_\alpha$. We assume that these contours go first from the corresponding infinity along the part of the adjoint cycle $\mathcal{B}_\alpha$ that lies in the sheet $\alpha_\pm$ until it reaches the intersection point P_α; after this point, we integrate along the cycle $\mathcal{A}_\alpha$ towards the final point x (see Fig. 4.1)

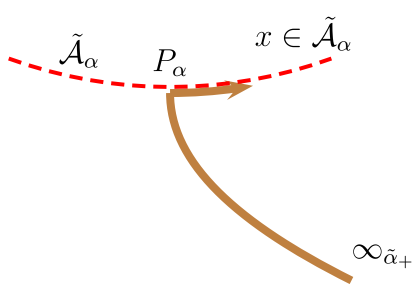


Figure 5: The path of integration w.r.t. the variable x' in the expression for the recursion kernel $K(\overset{\alpha}{x}, y)$.

To obtain the domain of the function $\widehat{K}(\overset{\alpha}{x}, z)$, we slightly deform the contours of integration over edges of the $\widetilde{\mathcal{A}}$-cycles as shown in Fig. 4.1; then, for the variable z lying in the domain that is "inner" w.r.t. integrations from infinities for all the functions $\psi_{\gamma_\pm}$, that is, for the domain that is separated from all the infinities $\infty_{\gamma_\pm}$ by the drawn apart edges of the $\widetilde{\mathcal{A}}$-cycles, the kernel $\widehat{K}(\overset{\alpha}{x}, z)$ is well defined (and it develops logarithmic cuts if we push the variable z through the boundary of the sheet S_α).

We now need to describe analytic properties of these functions.

For a fixed x, the kernel $\widehat{K}(\overset{\alpha}{x}, z)$ is defined for z in the cross-hatched domain in Fig. 3.1.1.

Taking an integration path between ∞_α and x we obtain that $\widehat{K}(\overset{\alpha}{x}, z)$ is defined for z outside this path. Across the path $]\infty_\alpha, x]$, $\widehat{K}(\overset{\alpha}{x}, z)$ has a discontinuity w.r.t. the

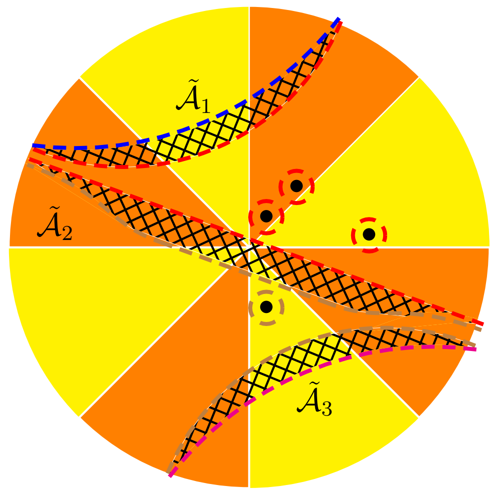


Figure 6: The domain of variable z (crosshatched) in (4.1) (we slightly deform the $\widetilde{\mathcal{A}}$-cycle integrals).

argument z:

$$\delta_z \widehat{K}(\overset{\alpha}{x}, z) = \frac{2i\pi}{\hbar}\, \frac{\psi_\alpha^2(z)}{\psi_\alpha^2(x)}. \tag{4.3}$$

The similar statement is true for $C_\alpha(z)$: when z cross the line of the cycle $\mathcal{A}_\beta$, we have

$$\delta_z C_\beta(z) = \frac{2i\pi\ \psi_{\beta_\pm}^2(z)}{\hbar} \int_z^{\infty_{\tilde{\beta}_\pm}} \frac{dx''}{\psi_{\beta_\pm}^2(x'')}. \tag{4.4}$$

We now define the recursion kernel $K(\overset{\alpha}{x}, z)$, which is the main ingredient of our construction.

Definition 4.1 *The recursion kernel $K(\overset{\alpha}{x}, z)$ reads*

$$K(\overset{\alpha}{x}, z) = \widehat{K}(\overset{\alpha}{x}, z) - \sum_{j=1}^{d-1} v_j(\overset{\alpha}{x}) C_j(z). \tag{4.5}$$

It is defined for z in the cross-hatched domain in Fig. 4.1.

Theorem 4.1 *The kernel K has the following properties:*

- *For a given z,*

$$K(\overset{\alpha}{x}, z) \sim O(x^{-2}) \tag{4.6}$$

  when $x \to \infty$ in all sectors (if the function $\psi_\alpha(x)$ increases in all the sectors except S_α).

- *The normalization condition reads*

$$\oint_{\mathcal{A}_j} K(x,z)dx = 0, \quad j = 1, \dots, d-1. \tag{4.7}$$

- $K(\overset{\alpha}{x}, z)$ *has double poles with zero residues at the zeros* $s_j^{(\alpha)}$ *of* ψ_α*.*

Theorem 4.2 *We have in all sectors at infinity :*

$$K(\overset{\alpha}{x}, z) \underset{z\to\infty}{\sim} O(z^{-d}). \tag{4.8}$$

More precisely we have:

$$K(\overset{\alpha}{x}, z) \sim -\sum_{k=d-1}^{\infty} \frac{K_k(\overset{\alpha}{x})}{z^{k+1}} \tag{4.9}$$

with

$$\widehat{K}_k(\overset{\alpha}{x}) = \frac{1}{\hbar\psi_\alpha^2(x)} \int_{\infty_\alpha}^{x} x'^k\, \psi_\alpha^2(x')\, dx', \tag{4.10}$$

and

$$K_k(\overset{\alpha}{x}) = \widehat{K}_k(\overset{\alpha}{x}) - \sum_{j=1}^{g} v_j(\overset{\alpha}{x}) \oint_{\mathcal{A}_j} \widehat{K}_k(x')\, dx'. \tag{4.11}$$

proof:

We can expand $\widehat{K}(\overset{\alpha}{x}, z)$ as

$$\widehat{K}(\overset{\alpha}{x}, z) \sim -\sum_{k=0}^{\infty} \frac{\widehat{K}_k(\overset{\alpha}{x})}{z^{k+1}} \tag{4.12}$$

where

$$\widehat{K}_k(\overset{\alpha}{x}) = \frac{1}{\hbar\psi_\alpha^2(x)} \int_{\infty_\alpha}^{x} x'^k\, \psi_\alpha^2(x')\, dx', \tag{4.13}$$

and therefore

$$K_k(\overset{\alpha}{x}) = \widehat{K}_k(\overset{\alpha}{x}) - \sum_{\alpha=1}^{g} v_\alpha(\overset{\alpha}{x}) \oint_{\mathcal{A}_\alpha} \widehat{K}_k(x')\, dx'. \tag{4.14}$$

For $k \leq d-2$, $(x')^k$ can be presented as a linear combinations of $h_j(x')$,

$$x'^k = \sum_{\beta=1}^{d-1} b_{k,\beta}\, h_\beta(x'), \tag{4.15}$$

and from the normalization condition, we immediately obtain that

$$\oint_{\mathcal{A}_\alpha} \widehat{K}_k(x')\, dx' = b_{k,\alpha}, \tag{4.16}$$

and therefore $K_k(x) = 0$ for $k \leq d-2$, which implies that

$$K(x,z) = O(z^{-d}). \tag{4.17}$$

□

4.2 Third kind differential: the kernel $G(\overset{\alpha}{x},\overset{\beta}{z})$

The second important kernel to define is the equivalent of the third kind differential.

Definition 4.2 *The kernel $G(\overset{\alpha}{x},\overset{\beta}{z})$ is*

$$G(\overset{\alpha}{x},\overset{\beta}{z}) = -\hbar\,\psi_\beta^2(z)\,\partial_z\,\frac{K(\overset{\alpha}{x},z)}{\psi_\beta^2(z)} = 2\hbar\frac{\psi'_\beta(z)}{\psi_\beta(z)}K(\overset{\alpha}{x},z) - \hbar\partial_z\,K(\overset{\alpha}{x},z) \tag{4.18}$$

Integration by parts gives

$$\begin{aligned} G(\overset{\alpha}{x},\overset{\beta}{z}) \;=\; & -\frac{1}{x-z} + \frac{2}{\psi_\alpha^2(x)}\int_{\infty_\alpha}^{x}\frac{dx'}{x'-z}\,\psi_\alpha^2(x')\left(\frac{\psi'_\alpha(x')}{\psi_\alpha(x')} - \frac{\psi'_\beta(z)}{\psi_\beta(z)}\right) \\ & -\hbar\sum_{j=1}^{d-1} v_j(\overset{\alpha}{x})\,\psi_\beta^2(z)\partial_z\,\frac{C_j(z)}{\psi_\beta^2(z)}. \end{aligned} \tag{4.19}$$

In what follows we often are in the situation when we take two integration contours, $\mathcal{C}_{D_x}$ and $\mathcal{C}_{D_z}$, and must interchange the order of integration (or the order in which these two contours intersect the $\mathcal{B}$-cycles). From the definition of the $\mathcal{A}$-cycles, it is then obvious that we must interchange the variables x and z within the same sector, so we need permutation relations for $G(\overset{\alpha}{x},\overset{\beta}{z})$ with $\alpha=\beta$. Then, as $x\to z$, we find that $G(\overset{\alpha}{x},\overset{\alpha}{z}) \sim \frac{1}{z-x}$, i.e., there is a simple pole with the unit residue at $z=x$. Because the combination $\frac{1}{x'-z}\left(\frac{\psi'_\alpha(x')}{\psi_\alpha(x')} - \frac{\psi'_\alpha(z)}{\psi_\alpha(z)}\right)$ is regular at $x'=z$, interchanging the order of integration over $\mathcal{C}_{D_x}$ and $\mathcal{C}_{D_z}$ then just gives the residue at $z=x$; no logarithmic cut takes place.

Theorem 4.3 *$G(\overset{\alpha}{x},\overset{\beta}{z})$ is an analytical function of x with a simple pole at $x=z$ with residue -1 for $\alpha=\beta$ and with double poles at the $s_j^{(\alpha)}$'s (zeros of $\psi_\alpha(x)$) with vanishing residues, and possibly an essential singularity at ∞.*

$G(\overset{\alpha}{x},\overset{\beta}{z})$ is an analytical function of z, with a simple pole at $z=x$ with residue $+1$ for $\alpha=\beta$, with simple poles at $z=s_j^{(\beta)}$, and with a discontinuity across $\mathcal{A}_\gamma$-cycles with $\gamma=1,\dots,g$ (this discontinuity has opposite signs depending on which line of the cycle $\mathcal{A}_\gamma$, γ_+ or γ_-, we cross; no discontinuity takes place when crossing the last cycle $\widetilde{\mathcal{A}}_n$):

$$\delta_z G(\overset{\alpha}{x},\overset{\beta_\pm}{z}) = \mp 2i\pi\, v_\beta(\overset{\alpha}{x}). \tag{4.20}$$

We also have

$$\oint_{\mathcal{A}_\alpha} G(x,\overset{\beta}{z})\,dx = 0. \tag{4.21}$$

proof:

All the discontinuities of $K(\overset{\alpha}{x}, z)$ except the one for $C_j(z)$ (4.4) are proportional to $\psi_\alpha^2(z)$ and vanish in (4.18). The discontinuity of $C_j(z)$ gives $\mp 2\pi i$, and thus, the discontinuity of $G(\overset{\alpha}{x}, \overset{\beta}{z})$ is $\delta_z G(\overset{\alpha}{x}, \overset{\beta\pm}{z}) = \mp 2i\pi\, v_\beta(\overset{\alpha}{x})$.

Since $K(\overset{\alpha}{x}, z)$ is regular when $z = s_j^{(\beta)}$, then it is clear that $G(\overset{\alpha}{x}, \overset{\beta}{z})$ has simple poles at $z = s_j^{(\beta)}$, with residue $-2\hbar K(x, s_j^{(\beta)})$.

In the variable x, $K(\overset{\alpha}{x}, z)$ has double poles at $x = s_j^{(\alpha)}$ without residue, and this property holds for $G(\overset{\alpha}{x}, \overset{\beta}{z})$ as well.

The vanishing $\mathcal{A}$-integral property follows immediately from (4.7). $\square$

Theorem 4.4

$$G(\overset{\alpha}{x}, \overset{\beta}{z}) = O(1/x^2) \tag{4.22}$$

when $x \to \infty_\alpha$ for any α.

At large z in the sector S_γ we have

$$\lim_{z\to\infty_\gamma} G(\overset{\alpha}{x}, \overset{\beta}{z}) = G(\overset{\alpha}{x}, \infty_\beta) = \eta_{\gamma,\beta}\; t_{d+1}\, K_{d-1}(\overset{\alpha}{x}) \tag{4.23}$$

where $\eta_{\gamma,\beta} = \pm 1$ depending on the asymptotic behavior of $\psi_\beta \sim e^{\pm V/2\hbar}$ in the sheet S_γ.

proof:

The large x behavior follows from theorem 4.1. The large z behavior is given by theorem 4.2, i.e. $G(\overset{\alpha}{x}, \overset{\beta}{z}) \sim \pm V'(z) K(\overset{\alpha}{x}, z) \sim \pm\, t_{d+1} K_{d-1}(\overset{\alpha}{x})$. The sign depends on the behavior of the solution in this sector. $\square$

4.3 The Bergman kernel $B(\overset{\alpha}{x}, \overset{\beta}{z})$

In classical algebraic geometry, the Bergman kernel is the fundamental second kind bi-differential, it is the derivative of the 3rd kind differential. Using the same definition as in [11], we define:

$$B(\overset{\alpha}{x}, \overset{\beta}{z}) = -\frac{1}{2}\,\partial_z\, G(\overset{\alpha}{x}, \overset{\beta}{z}). \tag{4.24}$$

We call the kernel B the "quantum" Bergman kernel.

Theorem 4.5 *$B(\overset{\alpha}{x}, \overset{\beta}{z})$ is an analytical function of x. When $\alpha = \beta$, it has a double pole at $x = z$ in the both variables x and z with no residue, and it has double poles in x and in z at the respective zeros $s_j^{(\alpha)}$ and $s_j^{(\beta)}$ with vanishing residues, and possibly an essential singularity at ∞. The discontinuity across $\mathcal{A}$-cycles that was present in the kernel G disappears upon differentiation, so $B(\overset{\alpha}{x}, \overset{\beta}{z})$ is defined analytically in the whole complex plane.*

proof:

These properties follow from those of $G(\overset{\alpha}{x},\overset{\beta}{z})$ of theorem 4.3. In particular, the only discontinuity of $G(\overset{\alpha}{x},\overset{\beta}{z})$ is along the $\mathcal{A}$-cycles, and it is independent of z, therefore $B(\overset{\alpha}{x},\overset{\beta}{z})$ has no discontinuity there. $\square$

4.3.1 Properties of the Bergman kernel

Theorem 4.6

$$B(\overset{\alpha}{x},\overset{\beta}{z}) = O(1/x^2) \tag{4.25}$$

when $x \to \infty$ in all sectors, and

$$B(\overset{\alpha}{x},\overset{\beta}{z}) = O(1/z^2) \tag{4.26}$$

when $z \to \infty$ in all sectors.

proof:

Follows from the large x and z behaviors of $G(\overset{\alpha}{x},\overset{\beta}{z})$. $\square$

Theorem 4.7 *The kernel B satisfies the loop equations:*

$$\left(2\frac{\psi'_\alpha(x)}{\psi_\alpha(x)} + \partial_x\right)\left(B(\overset{\alpha}{x},\overset{\beta}{z}) - \frac{1}{2(x-z)^2}\right) + \partial_z \frac{\frac{\psi'_\alpha(x)}{\psi_\alpha(x)} - \frac{\psi'_\beta(z)}{\psi_\beta(z)}}{x-z} = P_2^{(0)}(x,\overset{\beta}{z}), \tag{4.27}$$

where $P_2^{(0)}(x,\overset{\beta}{z})$ is a polynomial in x of degree at most $d-2$, and

$$\left(2\frac{\psi'_\beta(z)}{\psi_\beta(z)} + \partial_z\right)\left(B(\overset{\alpha}{x},\overset{\beta}{z}) - \frac{1}{2(x-z)^2}\right) + \partial_x \frac{\frac{\psi'_\alpha(x)}{\psi_\alpha(x)} - \frac{\psi'_\beta(z)}{\psi_\beta(z)}}{x-z} = \widetilde{P}_2^{(0)}(\overset{\alpha}{x},z) \tag{4.28}$$

where $\widetilde{P}_2^{(0)}(\overset{\alpha}{x},z)$ is a polynomial in z of degree at most $d-2$.

proof:

We begin with proving the first loop equation for $B(\overset{\alpha}{x},\overset{\beta}{z})$. We define:

$$\widehat{B}(\overset{\alpha}{x},\overset{\beta}{z}) = \frac{1}{2}\,\partial_z\left(2\frac{\psi'_\beta(z)}{\psi_\beta(z)} - \partial_z\right)\widehat{K}(\overset{\alpha}{x},z) \tag{4.29}$$

that is, we have

$$B(\overset{\alpha}{x},\overset{\beta}{z}) = \widehat{B}(\overset{\alpha}{x},\overset{\beta}{z}) - \sum_{j=1}^{d-1} v_j(\overset{\alpha}{x}) \oint_{\mathcal{A}_j} \widehat{B}(x'',\overset{\beta}{z})dx''. \tag{4.30}$$

Since $(2\frac{\psi'_\alpha(x)}{\psi_\alpha(x)} + \partial_x)\, v_j(\overset{\alpha}{x}) = h_j(x)$ is itself a polynomial of degree $\leq d-2$, it suffices to prove Eq. (4.27) for $\widehat{B}(\overset{\alpha}{x},\overset{\beta}{z})$.

We have

$$\left(2\frac{\psi'_\alpha(x)}{\psi_\alpha(x)}+\partial_x\right)\,\widehat{B}(\overset{\alpha}{x},\overset{\beta}{z}) \;=\; \frac{1}{2}\,\partial_z\,\left(2\frac{\psi'_\beta(z)}{\psi_\beta(z)}-\partial_z\right)\frac{1}{x-z}$$
$$\;=\; -\frac{1}{(x-z)^3}+\partial_z\,\frac{\psi'_\beta(z)}{\psi_\beta(z)(x-z)} \tag{4.31}$$

and therefore:

$$\left(2\frac{\psi'_\alpha(x)}{\psi_\alpha(x)}+\partial_x\right)\,\left(\widehat{B}(\overset{\alpha}{x},\overset{\beta}{z})-\frac{1}{2(x-z)^2}\right)+\partial_z\left(\frac{\frac{\psi'_\alpha(x)}{\psi_\alpha(x)}-\frac{\psi'_\beta(z)}{\psi_\beta(z)}}{x-z}\right)=0 \tag{4.32}$$

This proves Eq. (4.27) with

$$P_2^{(0)}(x,\overset{\beta}{z})=-\sum_{j=1}^{\mathfrak{g}}h_j(x)\,\oint_{\mathcal{A}_j}\widehat{B}(x'',\overset{\beta}{z})dx''. \tag{4.33}$$

We now prove the second loop equation for $B(\overset{\alpha}{x},\overset{\beta}{z})$. We have

$$\left(2\frac{\psi'_\beta(z)}{\psi_\beta(z)}+\partial_z\right)\,\widehat{B}(\overset{\alpha}{x},\overset{\beta}{z})=\frac{1}{2}\left(2\frac{\psi'_\beta(z)}{\psi_\beta(z)}+\partial_z\right)\,\partial_z\,\left(2\frac{\psi'_\beta(z)}{\psi_\beta(z)}-\partial_z\right)\,\widehat{K}(\overset{\alpha}{x},z), \tag{4.34}$$

where the operator $\widehat{U}(z)\equiv\frac{1}{2}\left(2\frac{\psi'_\beta(z)}{\psi_\beta(z)}+\partial_z\right)\,\partial_z\,\left(2\frac{\psi'_\beta(z)}{\psi_\beta(z)}-\partial_z\right)$ reads

$$\widehat{U}(z)=-\frac{1}{2}\,\partial_z^3+\frac{2}{\hbar^2}U(z)\partial_z+\frac{1}{\hbar^2}U'(z) \tag{4.35}$$

and is therefore independent on the solution $\psi_\beta(z)$ we started with. This is the Gelfand–Dikii operator [14]. We then have

$$\left(2\frac{\psi'_\beta(z)}{\psi_\beta(z)}+\partial_z\right)\,\widehat{B}(\overset{\alpha}{x},\overset{\beta}{z})=\frac{1}{\psi_\alpha^2(x)}\int_{\infty_\alpha}^{x}\psi_\alpha^2(x')\,dx'\left(-\frac{3}{(x'-z)^4}+\frac{2U(z)}{(x'-z)^2}+\frac{U'(z)}{x'-z}\right) \tag{4.36}$$

We integrate the first term by parts three times introducing $Y_\alpha(x)=\psi'_\alpha(x)/\psi_\alpha(x)$ (and exploiting that $Y'_\alpha+Y_\alpha^2=U$). The result reads

$$\left(2\frac{\psi'_\beta(z)}{\psi_\beta(z)}+\partial_z\right)\,\widehat{B}(\overset{\alpha}{x},\overset{\beta}{z})=\frac{1}{(x-z)^3}-\frac{\partial}{\partial x}\frac{Y_\alpha(x)}{x-z}+$$
$$+\frac{1}{\psi_\alpha^2(x)}\int_{\infty_\alpha}^{x}\psi_\alpha^2(x')\,dx'\left(2\frac{U(z)-U(x')}{(x'-z)^2}+\frac{U'(z)+U'(x')}{x'-z}\right) \tag{4.37}$$

This implies that

$$\left(2\frac{\psi'_\beta(z)}{\psi_\beta(z)}+\partial_z\right)\,\left(\widehat{B}(\overset{\alpha}{x},\overset{\beta}{z})-\frac{1}{2(x-z)^2}\right)+\frac{\partial}{\partial x}\,\frac{Y_\alpha(x)-Y_\beta(z)}{x-z}$$

$$= \frac{1}{\psi_\alpha^2(x)} \int_{\infty_\alpha}^{x} \psi_\alpha^2(x')\,dx' \Big(2\frac{U(z)-U(x')}{(x'-z)^2} + \frac{U'(z)+U'(x')}{x'-z}\Big), \tag{4.38}$$

and the obtained expression is obviously a polynomial in z. The expression in the brackets in (4.38) is a skew-symmetric polynomial in x' and z of degree not higher than $2d-2$. Moreover, all the terms with $(x')^k$ with $k \le d-2$ become upon integration linear combinations of $v_j(\overset{\alpha}{x})$ and vanish identically when we apply the projection to the subspace of zero $\mathcal{A}$-cycle integrals. So, the minimal power of x' that contributes to the answer is $(x')^{d-1}$. But there is no term $(x')^{d-1}z^{d-1}$ in the brackets because it contradicts the skew-symmetricity. The first nonzero term that might contribute is proportional to $(x')^{d-1}z^{d-2}$, which obviously means that the obtained polynomial $\widetilde{P}_2^{(0)}(\overset{\alpha}{x}, z)$ has the maximum degree at most $d-2$ in z. $\square$

Theorem 4.8 *We have for every $\alpha = 1, \dots, g$:*

$$\oint_{\mathcal{A}_i} B(x, \overset{\beta}{z})\,dx = 0, \qquad \oint_{\mathcal{A}_j} B(\overset{\alpha}{x}, z)\,dz = 0 \tag{4.39}$$

and

$$\oint_{\mathcal{B}_j} B(\overset{\alpha}{x}, z)\,dz = 2i\pi v_j(\overset{\alpha}{x}). \tag{4.40}$$

proof:

The vanishing of $\mathcal{A}$-cycle integrals in the x variable is by construction. For the z variable, we have

$$\begin{aligned}
\oint_{\mathcal{A}_\beta} B(\overset{\alpha}{x}, z)dz &= \int_{\infty_{\widetilde{\beta}_-}}^{\infty_{\widetilde{\beta}_+}} (B(\overset{\alpha}{x}, \overset{\beta_+}{z} - B(\overset{\alpha}{x}, \overset{\beta_-}{z})dz \\
&= -\frac{1}{2}\Big(G(\overset{\alpha}{x}, \overset{\beta_+}{\infty}_{\widetilde{\beta}_+}) - G(\overset{\alpha}{x}, \overset{\beta_-}{\infty}_{\widetilde{\beta}_+}) - G(\overset{\alpha}{x}, \overset{\beta_+}{\infty}_{\widetilde{\beta}_-}) + G(\overset{\alpha}{x}, \overset{\beta_-}{\infty}_{\widetilde{\beta}_-})\Big)
\end{aligned} \tag{4.41}$$

and the asymptotic conditions in all four cases for the function G in the second line are the same, so from theorem 4.4 we conclude that the result is zero.

We begin with the integral over a cycle $\widetilde{\mathcal{B}}_\beta$:

$$\begin{aligned}
\oint_{\widetilde{\mathcal{B}}_\beta} B(\overset{\alpha}{x}, y)dy &= \text{jump of } G(\overset{\alpha}{x}, y) \text{ on } \widetilde{\mathcal{A}}_\beta \\
&\quad -\frac{1}{2}\Big(G(\overset{\alpha}{x}, \overset{\beta_+}{\infty}_{\beta_+}) - G(\overset{\alpha}{x}, \overset{\beta_-}{\infty}_{\beta_+}) - G(\overset{\alpha}{x}, \overset{\beta_+}{\infty}_{\beta_-}) + G(\overset{\alpha}{x}, \overset{\beta_-}{\infty}_{\beta_-})\Big) \\
&= 2\pi i(1-\delta_{\beta,d})v_\beta(\overset{\alpha}{x}) + 2K_{d-1}(\overset{\alpha}{x}),
\end{aligned} \tag{4.42}$$

where we have used again the asymptotic conditions from Theorem 4.4. This formula implies that if we perform the integration $\oint_{\mathcal{B}_j}$ for $j = 1, \dots, d-1$, which is the difference of integrals over the cycles $\widetilde{\mathcal{B}}_j$ and $\widetilde{\mathcal{B}}_d$, we obtain the formula (4.40). $\square$

The key property of the Bergman kernel is provided by the following theorem.

Theorem 4.9 $B(\overset{\alpha}{x},\overset{\beta}{z})$ *is symmetric,*

$$B(\overset{\alpha}{x},\overset{\beta}{z}) = B(\overset{\beta}{z},\overset{\alpha}{x}) \tag{4.43}$$

proof:

The proof uses that $B(\overset{\alpha}{x},\overset{\beta}{z})$ satisfies the loop equation in the two variables. We have:

$$\begin{aligned}
& \left(2\frac{\psi'_\beta(z)}{\psi_\beta(z)}+\partial_z\right)\left(2\frac{\psi'_\alpha(x)}{\psi_\alpha(x)}+\partial_x\right)\left(B(\overset{\alpha}{x},\overset{\beta}{z})-\frac{1}{2(x-z)^2}\right) \\
= & \left(2\frac{\psi'_\beta(z)}{\psi_\beta(z)}+\partial_z\right)\left(P_2^{(0)}(x,\overset{\beta}{z})-\partial_z\frac{\frac{\psi'_\alpha(x)}{\psi_\alpha(x)}-\frac{\psi'_\beta(z)}{\psi_\beta(z)}}{x-z}\right) \\
= & \left(2\frac{\psi'_\alpha(x)}{\psi_\alpha(x)}+\partial_x\right)\left(\widetilde{P}_2^{(0)}(\overset{\alpha}{x},z)-\partial_x\frac{\frac{\psi'_\alpha(x)}{\psi_\alpha(x)}-\frac{\psi'_\beta(z)}{\psi_\beta(z)}}{x-z}\right)
\end{aligned} \tag{4.44}$$

We then have

$$\begin{aligned}
& \left(2\frac{\psi'_\beta(z)}{\psi_\beta(z)}+\partial_z\right)P_2^{(0)}(x,\overset{\beta}{z})-\left(2\frac{\psi'_\alpha(x)}{\psi_\alpha(x)}+\partial_x\right)\widetilde{P}_2^{(0)}(\overset{\alpha}{x},z) \\
= & \left(2\frac{\psi'_\beta(z)}{\psi_\beta(z)}+\partial_z\right)\partial_z\frac{\frac{\psi'_\alpha(x)}{\psi_\alpha(x)}-\frac{\psi'_\beta(z)}{\psi_\beta(z)}}{x-z}-\left(2\frac{\psi'_\alpha(x)}{\psi_\alpha(x)}+\partial_x\right)\partial_x\frac{\frac{\psi'_\alpha(x)}{\psi_\alpha(x)}-\frac{\psi'_\beta(z)}{\psi_\beta(z)}}{x-z} \\
= & 2\frac{U(x)-U(z)}{(x-z)^2}-\frac{U'(x)+U'(z)}{x-z},
\end{aligned} \tag{4.45}$$

so that

$$\begin{aligned}
& (x-z)^2\left(2\frac{\psi'_\beta(z)}{\psi_\beta(z)}+\partial_z\right)P_2^{(0)}(x,\overset{\beta}{z})+2U(z)+(x-z)U'(z) \\
= & (x-z)^2\left(2\frac{\psi'_\alpha(x)}{\psi_\alpha(x)}+\partial_x\right)\widetilde{P}_2^{(0)}(\overset{\alpha}{x},z)+2U(x)+(z-x)U'(x) \\
\overset{\text{def}}{=} & R(x,z).
\end{aligned} \tag{4.46}$$

Here, the first line is a polynomial in x, whereas the second line is in turn a polynomial in z. Therefore, $R(x,z)$ is a polynomial in the both variables, of degree at most d in each variable. Moreover, we must have:

$$R(x,x) = 2U(x) \tag{4.47}$$

Therefore we must have:

$$R(x,z) = \frac{1}{\hbar^2}\left(\frac{1}{2}V'(x)V'(z)-\hbar\frac{V'(x)-V'(z)}{x-z}-P(x)-P(z)\right)+(x-z)^2\widetilde{R}(x,z) \tag{4.48}$$

where $\widetilde{R}(x,z)$ is a polynomial in the both variables of degree at most $d-2$ in each variable.

Putting this back into (4.46) and using the symmetry $x \leftrightarrow z$, we obtain

$$\left(2\frac{\psi'_\beta(z)}{\psi_\beta(z)} + \partial_z\right)(P_2^{(0)}(x, \overset{\beta}{z}) - \widetilde{P}_2^{(0)}(\overset{\beta}{z}, x)) = \widetilde{R}(x,z) - \widetilde{R}(z,x) \tag{4.49}$$

Then, we can decompose the r.h.s into the basis $h_i(x)h_j(z)$,

$$\widetilde{R}(x,z) - \widetilde{R}(z,x) = \sum_{i,j=1}^{d-1} (\widetilde{R}_{i,j} - \widetilde{R}_{j,i}) h_i(x) h_j(z) \tag{4.50}$$

Applying the integral operator

$$f(z) \mapsto \frac{1}{\psi_\beta^2(z)} \int_{\infty_\beta}^{z} \psi_\beta^2(z') f(z') dz' \tag{4.51}$$

to the differential equation (4.49), we obtain

$$P_2^{(0)}(x, \overset{\beta}{z}) - \widetilde{P}_2^{(0)}(\overset{\alpha}{z}, x) = \sum_{i,j=1}^{d-1} (\widetilde{R}_{i,j} - \widetilde{R}_{j,i}) h_i(x)\, v_j(\overset{\beta}{z}) + A_1(x) \tag{4.52}$$

where $A_1(x)$ is some integration constant.

Then using the loop equations (4.7) we find by substraction that:

$$\left(2\frac{\psi'_\alpha(x)}{\psi(x)} + \partial_x\right)\left(B(\overset{\alpha}{x}, \overset{\beta}{z}) - B(\overset{\beta}{z}, \overset{\alpha}{x})\right) = P_2^{(0)}(x, \overset{\beta}{z}) - \widetilde{P}_2^{(0)}(\overset{\beta}{z}, x) \tag{4.53}$$

and applying the integral operator (4.51) w.r.t. the variable x in the sheet S_α, we obtain

$$B(\overset{\alpha}{x}, \overset{\beta}{z}) - B(\overset{\beta}{z}, \overset{\alpha}{x}) = \sum_{i,j=1}^{d-1} (\widetilde{R}_{i,j} - \widetilde{R}_{j,i}) v_i(\overset{\alpha}{x})\, v_j(\overset{\beta}{z}) + A(x) + \widetilde{A}(z), \tag{4.54}$$

with $A(x)$ and $\widetilde{A}(z)$ the integration constants.

Next, the large x and large z behavior of B imply that $A(x) = \widetilde{A}(z) = 0$, and therefore

$$B(\overset{\alpha}{x}, \overset{\beta}{z}) - B(\overset{\beta}{z}, \overset{\alpha}{x}) = \sum_{i,j} (\widetilde{R}_{i,j} - \widetilde{R}_{j,i})\, v_i(\overset{\alpha}{x}) v_j(\overset{\beta}{z}). \tag{4.55}$$

Then, using theorem 4.8

$$\oint_{\mathcal{A}_i} B(x, \overset{\beta}{z}) dx = \oint_{\mathcal{A}_j} B(\overset{\alpha}{x}, z) dz = 0 \;\forall\; i, j, \tag{4.56}$$

we obtain that

$$\widetilde{R}_{i,j} = \widetilde{R}_{j,i} \;\forall i, j, \tag{4.57}$$

which completes the proof of the symmetricity of the Bergman kernel. $\square$

We see therefore that our "quantum Bergman kernel" enjoys all the features of the standard Bergman kernel associated with a Riemann surface: it is symmetric, has no discontinuities, and possesses the double pole with no residue at the coinciding arguments (which corresponds in our case to the coinciding arguments on the *same* sheet S_k). Using all these kernels we can then generalize the recursion of [9, 6, 11] defining the correlation functions (see the next section).

4.4 Meromorphic forms and the Riemann bilinear identity

Definition 4.3 *A meromorphic form* $\mathcal{R}(\overset{\alpha}{x})$ *reads*

$$\mathcal{R}(\overset{\alpha}{x}) = \frac{1}{\hbar\psi_\alpha^2(x)} \int_{\infty_\alpha}^{x} r(x')\, \psi_\alpha^2(x')\, dx' \tag{4.58}$$

where $r(x)$ *is a rational function of* x*, which behaves at most like* $O(x^{d-2})$ *at large* x *and whose poles* r_i *are such that*

$$\operatorname*{Res}_{x\to r_i} \psi_\alpha^2(x)\, r(x) = 0. \tag{4.59}$$

The holomorphic forms $v_j(\overset{\alpha}{x})$, the kernels $G(\overset{\alpha}{x},\overset{\beta}{z})$ and $B(\overset{\alpha}{x},\overset{\beta}{z})$ are meromorphic forms of x.

A meromorphic form $\mathcal{R}(\overset{\alpha}{x})$ defined by (4.58) has poles at $x = r_i$, the poles of $r(x)$, with degree one less than that of $r(x)$, it has double poles at the $s_i^{(\alpha)}$ with vanishing residues, and it behaves like $O(x^{-2})$ in all sectors (having also an accumulation of poles along the half-lines L_i of accumulations of zeroes of ψ_α). Note that the integrals $\oint_{\mathcal{A}_\alpha} \mathcal{R}(x)dx$ are well defined.

We then have the following theorem.

Theorem 4.10 Riemann bilinear identity

Consider a meromorphic form $\mathcal{R}(\overset{\alpha}{z})$*. For* z *in the sector* S_α *(outside the crosshatched domain in Fig. 4.1), we obtain the representation formula*

$$\begin{aligned}\mathcal{R}(\overset{\alpha}{z}) \;=\; & -\sum_\beta \sum_{r_i\in S_\beta} \operatorname*{Res}_{r_i} G(\overset{\alpha}{z},\overset{\beta}{y})\mathcal{R}(\overset{\beta}{y})dy - \sum_\beta \sum_{s_k^{(\beta)}\in S_\beta} \operatorname*{Res}_{s_k^{(\beta)}} G(\overset{\alpha}{z},\overset{\beta}{y})\mathcal{R}(\overset{\beta}{y})dy \\ & + \sum_{j=1}^{\mathfrak{g}} v_j(\overset{\alpha}{z}) \oint_{\mathcal{A}_j} \mathcal{R}(y)\, dy. \end{aligned} \tag{4.60}$$

proof:

We begin with the integral over $\mathcal{C}_D$ (Fig. 3.1) of $G(\overset{\alpha}{z},y)\mathcal{R}(y)dy$:

$$\int_{\mathcal{C}_D} G(\overset{\alpha}{z},y)\mathcal{R}(y)dy = \sum_\beta \int_{\infty_{\beta-1}}^{\infty_{\beta+1}} G(\overset{\alpha}{z},\overset{\beta}{y})\mathcal{R}(\overset{\beta}{y})dy. \tag{4.61}$$

This integral is identically zero because of asymptotic conditions $G(\overset{\alpha}{z},\overset{\beta}{y}) \to 1/y$ as $y \to \infty_\beta$ and $\mathcal{R}(\overset{\beta}{y}) \sim 1/y^2$ as $y \to \infty_\beta$ and by the fact that no accumulation of zeros occurs for the function $\mathcal{R}(\overset{\beta}{y})$ on the boundaries between the sectors S_β and $S_{\beta\pm 1}$. We then push the integration contours through the complex plane towards the $\widetilde{\mathcal{A}}$-cycles as in Fig. 3.1.1; the residues at the points r_i and $s_k^{(\beta)}$ give the first line of (4.60), the residue at the point $x = y$ in the sector S_α gives the left-hand side because we have from Theorem 4.3 that $G(\overset{\alpha}{z},\overset{\alpha}{y}) = 1/(z-y) + \text{reg.}$; it remains to consider the integrals along the $\widetilde{\mathcal{A}}$-cycles. For this, note that our integrations over y are outer w.r.t. the integrations over the variable x along $\mathcal{A}$-cycles in the formula (4.2) for the factors C_j, and when we push the integration over y through that for x, we have the discontinuity of $G(\overset{\alpha}{z},\overset{\beta\pm}{y})$, which is equal to $2\pi i v_j(\overset{\alpha}{z})$; no such discontinuity occurs for the cycle $\widetilde{\mathcal{A}}_d$. All these discontinuities are independent of y, so the contour integral of the product factorizes for each cycle $\mathcal{A}_j$ and gives the second line of (4.60).

To evaluate the remaining integrals inside the crosshatched domain in Fig. 4.1 recall that $G(\overset{\alpha}{x},\overset{\beta}{y}) = \psi_\beta^2(y)\,\partial_y\left(K(\overset{\alpha}{x},y)/\psi_\beta^2(y)\right)$, so integrating by parts we obtain

$$\int_{\infty_{\tilde\beta_-}}^{\infty_{\tilde\beta_+}} \Big(G(\overset{\alpha}{x},\overset{\beta_+}{z})\mathcal{R}(\overset{\beta_+}{z}) - G(\overset{\alpha}{x},\overset{\beta_-}{z})\mathcal{R}(\overset{\beta_-}{z})\Big)dz = -\int_{\infty_{\tilde\beta_-}}^{\infty_{\tilde\beta_+}} (K(\overset{\alpha}{x},z)r(z) - K(\overset{\alpha}{x},z)r(z))dz = 0, \tag{4.62}$$

and these contributions vanish for all the cycles $\mathcal{A}_\beta$. $\square$

5 Correlation functions. Diagrammatic representation

In this section, we define the sectorwise defined versions of the quantum correlation functions from paper I (deformations of "classical" correlation functions introduced in [9, 6, 12]). Our definitions are inspired from (non-Hermitian) eigenvalue models (see section 8), but they are valid as well in a general setting of an arbitrary Schrödinger equation.

5.1 Definition and properties of correlation functions

Definition 5.1 *We define the functions $W_n^{(h)}(\overset{\alpha_1}{x}_1,\dots,\overset{\alpha_n}{x}_n)$ called the* n-point correlation function of "genus" h *by the recursion*

$$W_1^{(0)}(\overset{\alpha}{x}) = \omega(\overset{\alpha}{x}), \qquad W_2^{(0)}(\overset{\alpha_1}{x}_1,\overset{\alpha_2}{x}_2) = B(\overset{\alpha_1}{x}_1,\overset{\alpha_2}{x}_2) \tag{5.1}$$

$$W_{n+1}^{(h)}(\overset{\alpha_0}{x}_0, J) = \oint_{\mathcal{C}_{D_x}} dx\, K(\overset{\alpha_0}{x}_0, x)\Big(\overline{W}_{n+2}^{(h-1)}(x,x,J) + \sum_{r=0}^{h}\sum_{I\subset J}{}' W_{|I|+1}^{(r)}(x,I)W_{n-|I|+1}^{(h-r)}(x,J/I)\Big) \tag{5.2}$$

where J and I are the collective notation for the variables ($J = \{x_1,\dots,x_n\}$), the symbol $\sum\sum'$ means that we exclude the terms ($r=0, I=\emptyset$), ($r=0, I=\{x_i\}$), ($r=h, I=J/\{x_i\}$), and ($r=h, I=J$), the integration over the contour $\mathcal{C}_{D_x}$ is defined in (3.1), and where

$$\overline{W}_n^{(h)}(\overset{\alpha_1}{x}_1,\dots,\overset{\alpha_n}{x}_n) = W_n^{(h)}(\overset{\alpha_1}{x}_1,\dots,\overset{\alpha_n}{x}_n) - \frac{\delta_{n,2}\delta_{h,0}\delta_{\alpha_1,\alpha_2}}{2(x_1-x_2)^2}. \tag{5.3}$$

Here the point x_0 is outside the integration contour $\mathcal{C}_{D_x}$ for the variable x and all the x_i are outside the $\mathcal{A}$-cycles of the projection integrals.

To shorten equations we introduce the notation

$$U_n^{(h)}(\overset{\alpha}{x}, J) = \overline{W}_{n+2}^{(h-1)}(\overset{\alpha}{x},\overset{\alpha}{x},J) + \sum_{r=0}^{h}\sum_{I\subset J}{}' \overline{W}_{|I|+1}^{(r)}(\overset{\alpha}{x},I)\overline{W}_{n-|I|+1}^{(h-r)}(\overset{\alpha}{x},J/I). \tag{5.4}$$

The main property of 5.1 is that these quantities solve the loop equations in the $1/N^2$-expansion. We also prove the following properties:

Theorem 5.1 *Each $W_n^{(h)}(\overset{\alpha_1}{x}_1,\dots,\overset{\alpha_n}{x}_n)$ with $2-2h-n<0$ is an analytical functions of all its arguments with poles only when $x_i \to s_j^{(\alpha_i)}$. It vanishes at least as $O(1/x_i^2)$ when $x_i \to \infty_{\alpha_i}$ and has no discontinuities across $\mathcal{A}$-cycles.*

Corollary 5.1 *We have that*

$$\int_{\mathcal{C}_{D_x}} W_{n+1}^{(h)}(x,J)dx = t_0\delta_{n,0}\delta_{h,0}.$$

proof:

We proceed by recursion on $2h+n$. The analyticity is obvious; the theorem is true for $W_2^{(0)}$. Assume it is true up to $2g+n$, we shall prove it for $W_{n+1}^{(g)}(x_0,x_1,\dots,x_n)$. To prove the asymptotic behavior, note that the definition 4.1 implies, first, that the term $\int_{\mathcal{C}_{D_y}}\widehat{K}(\overset{\alpha}{x},y)U_n^{(h)}(y,J)dy$ is of order of $1/(\psi_\alpha^2(x))\int_{\infty_\alpha}^{x}\psi_\alpha^2(x')dx'/(x')^2 \sim x^{-d-2}$, so the leading contribution comes from the terms proportional to $v_j(\overset{\alpha}{x}) \sim x^{-2}$, which completes the proof. $\square$

We also have the following simple lemma that follows from the corollary 5.1 and from the normalization conditions for the kernel $K(\overset{\alpha}{x},y)$.

Lemma 5.1 *For all $(n,h)\neq(0,0)$ we have*

$$\oint_{\widetilde{\mathcal{A}}_\alpha} W_{n+1}^{(h)}(x,J)dx = 0. \tag{5.5}$$

Now comes first of the main theorems.

Theorem 5.2 *For $2-2h-n<0$, the $W_n^{(h)}$ satisfy the loop equation. This means that the quantity*

$$\begin{aligned}
P_{n+1}^{(h)}(x;\overset{\alpha_1}{x}_1,\ldots,\overset{\alpha_n}{x}_n) &= \hbar\left(2\frac{\psi'_\alpha(x)}{\psi_\alpha(x)}+\partial_x\right)\overline{W}_{n+1}^{(h)}(\overset{\alpha}{x},\overset{\alpha_1}{x}_1,\ldots,\overset{\alpha_n}{x}_n) \\
&\quad+\sum_{r=0}^{h}{\sum_{I\subset J}}' \overline{W}_{|I|+1}^{(r)}(\overset{\alpha}{x},I)\overline{W}_{n-|I|+1}^{(h-r)}(\overset{\alpha}{x},J/I)+\overline{W}_{n+2}^{(h-1)}(\overset{\alpha}{x},\overset{\alpha}{x},J) \\
&\quad+\sum_j \partial_{x_j}\left(\frac{\overline{W}_n^{(h)}(\overset{\alpha}{x},J/\{x_j\})\delta_{\alpha,\alpha_j}-\overline{W}_n^{(h)}(\overset{\alpha_j}{x}_j,J/\{x_j\})}{(x-x_j)}\right) \qquad (5.6)
\end{aligned}$$

is a polynomial in the variable x, which is of degree at most $d-2$ and is independent on the choice of the sector S_α.

proof:

in appendix A □

Theorem 5.3 *Each $W_n^{(h)}$ is a symmetric function of all its arguments.*

proof:

in appendix C, with the special case of $W_3^{(0)}$ in appendix B. □

Theorem 5.4 *For $2-2h-n<0$, $W_n^{(h)}(\overset{\alpha_1}{x}_1,\ldots,\overset{\alpha_n}{x}_n)$ is homogeneous of degree $2-2h-n$:*

$$\begin{aligned}
&\left(\hbar\frac{\partial}{\partial\hbar}+\sum_{j=1}^{d+1}t_j\frac{\partial}{\partial t_j}+\sum_{i=1}^{\mathfrak{g}}\epsilon_i\frac{\partial}{\partial\epsilon_i}+t_0\frac{\partial}{\partial t_0}\right)W_n^{(h)}(\overset{\alpha_1}{x}_1,\ldots,\overset{\alpha_n}{x}_n) \\
&\quad=(2-2h-n)\,W_n^{(h)}(\overset{\alpha_1}{x}_1,\ldots,\overset{\alpha_n}{x}_n). \qquad (5.7)
\end{aligned}$$

proof:

Under a change $t_k\to\lambda t_k$, $\hbar\to\lambda\hbar$, $\epsilon_i\to\lambda\epsilon_i$, and $t_0\to\lambda t_0$ the Schrödinger equation remains unchanged, and thus ψ is unchanged. The kernel K is changed to K/λ and nothing else is changed. By recursion, $W_n^{(h)}$ is then changed by λ^{2-2h-n}. □

5.2 Diagrammatic representation

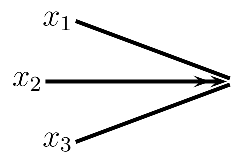
The diagrammatic representation for the correlation functions structurally coincides with the one for the correlation functions in one- and two-matrix models [9, 4, 6]. We introduce the three kinds of propagators

$K(\overset{\alpha}{x},y)$

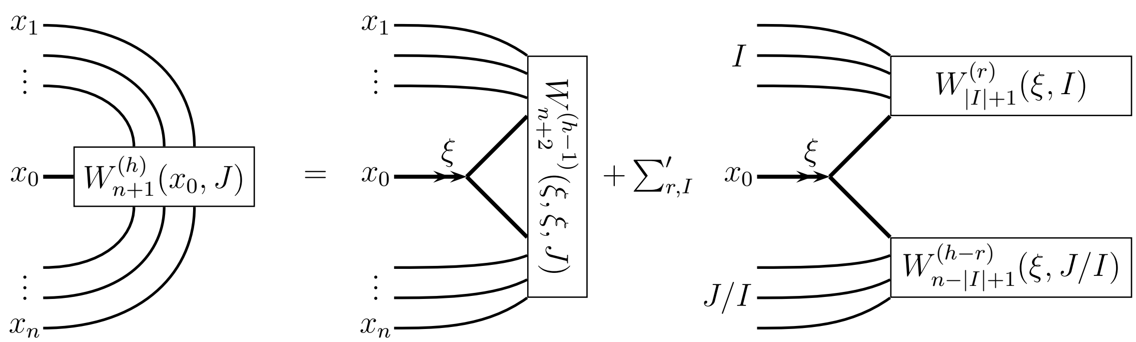
$G(\overset{\alpha}{x},\overset{\beta}{y})$

$B(\overset{\alpha}{x},\overset{\beta}{y})$

and assume the partial ordering from "infinity" to "$\mathcal{A}$-cycles" to be from left to the right in graphical expressions. We represent the terms $W_n^{(g)}(J)$ via the graphs with three-valent vertices; we assign its own variable ξ to every inner vertex and assume the integration w.r.t. this variable along the contour $\mathcal{C}_D$, the order of integration depends on which vertex is closer to the "$\mathcal{A}$-cycles": we begin with integrating at the innermost vertex. We also have n outer legs (one-valent vertices) corresponding to the points $\overset{\alpha_i}{x}_i$, $i = 1, \dots, n$. They are assumed to lie outside all the inner integrations. For example, the term $W_3^{(0)}(x_1, x_2, x_3)$ then has the form



whereas the recurrent relation (5.2) takes the form



We now formulate the diagrammatic technique for constructing the functions $W_n^{(h)}(J)$ for $n > 0$ and $2g - 2 + n > 0$. Formally it is the same as the one in [9, 6].

We comprise all the diagrams with the corresponding automorphism multipliers such that

- for $W_n^{(h)}(J)$ a diagram contains exactly n external legs and h loops;
- we segregate one variable, say, x_1, and take all the maximum connected rooted subtrees starting at the vertex x_1 and not going to any other external leg;
- we associate the directed propagators $K(x, y)$ with all the edges of the rooted subtree; the direction is always from the root to branches;
- all other propagators that comprise exactly h inner propagators and $n - 1$ remaining external legs are $B(x, y)$ if the vertices x and y are distinct and $\overline{B}(x, x)$ for the loop composed from the single propagator;

- each rooted subtree establishes the partial ordering on the set of three-valent vertices of the diagram; we allow the inner propagators $B(x, y)$ to connect *only* the comparable vertices (a vertex is comparable to itself).

6 Deformations

In this section, we consider the variations of correlation functions $W_n^{(g)}$ under infinitesimal variations of the Schrödinger potential $U(x)$ or $\hbar$. Infinitesimal variations of the resolvent $\omega(x)$ can be decomposed on the basis of "meromorphic forms" $v_k(\overset{\alpha}{x})$, $k = 1, \ldots$; we set these forms to be dual to special cycles with the duality kernel being the Bergman kernel. It turns out that the classical $\hbar = 0$ formulas retain their form for $\hbar \neq 0$.

6.1 Variation of the resolvent

We consider an infinitesimal polynomial variation:

$$U \to U + \delta U, \qquad \hbar \to \hbar + \delta\hbar.$$

Since $U = V'^2/4 - \hbar V''/2 - P$, we have

$$\delta U = \frac{V'}{2}\delta V' - \frac{\hbar}{2}\delta V'' - \frac{\delta\hbar}{2}\delta V'' - \delta P. \tag{6.1}$$

We can consider variations of $V'(z)$ w.r.t. the higher times t_k, $k = 1, \ldots$, as well:

$$\delta V'(x) = \sum_{k=1} \delta t_k \, x^{k-1}. \tag{6.2}$$

Then, for $k \leq d+1$, δP is of degree at most $d-1$ whereas for $k > d+1$ the polynomial δP has the degree at most $k-2$.

Computing $\delta\left(\psi'_\alpha(x)/\psi_\alpha(x)\right)$, we obtain

$$\delta\left(\psi'_\alpha(x)/\psi_\alpha(x)\right) = \frac{1}{\hbar^2\,\psi_\alpha^2(x)} \int_{\infty_\alpha}^{x} \psi_\alpha^2(x') \left(\delta U(x') - 2\,\frac{\delta\hbar}{\hbar}\,U(x')\right) dx', \tag{6.3}$$

and for $\omega(\overset{\alpha}{x}) = V'(x)/2 + \hbar\psi'_\alpha(x)/\psi_\alpha(x)$ we have

$$\delta\omega(\overset{\alpha}{x}) = \frac{\delta V'(x)}{2} + \delta\hbar\,\frac{\psi'_\alpha(x)}{\psi_\alpha(x)} + \frac{1}{\hbar\,\psi_\alpha^2(x)} \int_{\infty_\alpha}^{x} \psi_\alpha^2(x') \left(\delta U(x') - 2\,\frac{\delta\hbar}{\hbar}\,U(x')\right) dx'. \tag{6.4}$$

6.2 Variations w.r.t. "flat" coordinates

We choose a system of "flat" coordinates on our genus-$(d-1)$ manifold:

$$\epsilon_1, \ldots, \epsilon_{d-1}, t_0, t_1, \ldots . \tag{6.5}$$

6.2.1 Variations w.r.t. the filling fractions

For the filling fraction $\delta\epsilon_\alpha$ we have $\delta V' = 0$ and thus

$$\delta U(x) = -\delta P(x) \tag{6.6}$$

where $\deg \delta P \leq d-2$, so we can decompose it in the basis of h_α:

$$\delta P(x) = \sum_{\alpha'} c_{\alpha'}\, h_{\alpha'}. \tag{6.7}$$

So, from (6.4), we have

$$\delta\omega(x) = -\sum_{\alpha'} c_{\alpha'}\, v_{\alpha'}(x), \tag{6.8}$$

and because $2i\pi\epsilon_{\alpha'} = \oint_{\mathcal{A}_{\alpha'}} \omega$, we obtain

$$2i\pi\, \delta_{\alpha,\alpha'} = \oint_{\mathcal{A}_{\alpha'}} \delta\omega = -\sum_{\alpha''} \oint_{\mathcal{A}_{\alpha'}} c_{\alpha''}\, v_{\alpha''} = -c_{\alpha'} \tag{6.9}$$

Therefore $\delta U(x)/\delta_{\epsilon_\alpha} = 2i\pi h_\alpha(x)$ and

$$\delta_{\epsilon_\alpha}\omega(\overset{\beta}{x}) = 2i\pi\, v_\alpha(\overset{\beta}{x}) = \oint_{\mathcal{B}_\alpha} B(\overset{\beta}{x}, z)\, dz. \tag{6.10}$$

The flat coordinate ϵ_α is dual to the holomorphic form v_α, which is itself dual to the cycle $\mathcal{B}_\alpha$:

$$\epsilon_\alpha = \frac{1}{2i\pi} \oint_{\mathcal{A}_\alpha} \omega, \qquad \delta_{\epsilon_\alpha}\omega = 2i\pi\, v_\alpha = \oint_{\mathcal{B}_\alpha} B. \tag{6.11}$$

6.2.2 Variation w.r.t. t_0

We have

$$\delta U(x) = -\delta P(x) = -t_{d+1}\, x^{d-1} + Q(x), \tag{6.12}$$

where $\deg Q \leq d-2$. Using Eq. (6.4) we obtain

$$\delta\omega(\overset{\alpha}{x}) = \frac{1}{\psi_\alpha^2(x)} \int_{\infty_\alpha}^{x} (-t_{d+1} x'^{d-1} + Q(x'))\, \psi_\alpha^2(x')\, dx', \tag{6.13}$$

and the polynomial Q must be chosen such that $\oint_{\mathcal{A}_i} \delta\omega = 0$ We therefore have

$$\begin{aligned}
\delta\omega(\overset{\alpha}{x}) &= -\, t_{d+1}\, K_{d-1}(\overset{\alpha}{x}) \\
&= -t_{d+1} \left(\widehat{K}_{d-1}(\overset{\alpha}{x}) - \sum_{\beta=1}^{d-1} v_\beta(\overset{\alpha}{x}) \oint_{\mathcal{A}_\beta} \widehat{K}_{d-1}(x')\, dx' \right) \\
&= v_d(\overset{\alpha}{x}),
\end{aligned} \tag{6.14}$$

where

$$\widehat{K}_k(\overset{\alpha}{x}) = \frac{1}{\psi_\alpha^2(x)} \int_{\infty_\alpha}^{x} x'^k \, \psi_\alpha^2(x')\, dx', \tag{6.15}$$

and $K_k(\overset{\alpha}{x})$ is the kth term in the large z expansion of $K(\overset{\alpha}{x}, z) = -\sum_{k=0}^{\infty} \frac{K_k(\overset{\alpha}{x}, z)}{z^{k+1}}$ computed in theorem 4.2. From theorem 4.4 we have $G(x, \infty_\alpha) = \eta_\alpha t_{d+1}\, K_{d-1}(\overset{\alpha}{x})$. This shows that

$$\delta_{t_0}\omega(\overset{\alpha}{x}) = \frac{1}{2}\left(G(\overset{\alpha}{x}, \infty_{\tilde{d}_+}) - G(\overset{\alpha}{x}, \infty_{\tilde{d}_-})\right) = \int_{\infty_{\tilde{d}_-}}^{\infty_{\tilde{d}_+}} B(\overset{\alpha}{x}, z)\, dz. \tag{6.16}$$

The integral is taken here over the last, dth $\widetilde{B}$-cycle.

The flat coordinate t_0 is then dual to the 3rd kind meromorphic form $-2G(\overset{\alpha}{x}, \infty.)$, which is itself dual to the cycle $[\infty_{\tilde{d}_-}, \infty_{\tilde{d}_+}]$:

$$t_0 = \oint_{\mathcal{C}_D} \omega(z) dz, \qquad \delta_{t_0}\omega(\overset{\alpha}{x}) = \int_{\infty_{\tilde{d}_-}}^{\infty_{\tilde{d}_+}} B(\overset{\alpha}{x}, z)\, dz. \tag{6.17}$$

6.2.3 Variations w.r.t. $t_k, k = 1 \ldots$. The two-point correlation function.

Because

$$t_k = \oint_{\mathcal{C}_D} \hbar \psi'(z)/\psi(z) z^{-k} dz, \quad k = 0, 1, \ldots,$$

the conditions $\partial t_k/\partial t_r = \delta_{k,r}$ and $\partial t_k/\partial \epsilon_\beta = 0$ imply

$$\oint_{\mathcal{C}_D} \frac{\partial}{\partial t_r}\left(\hbar \frac{\psi'(z)}{\psi(z)}\right) z^{-k} dz = \delta_{k,r}; \qquad \oint_{\mathcal{A}_\beta} \frac{\partial}{\partial t_r}\left(\hbar \frac{\psi'(z)}{\psi(z)}\right) dz = 0, \tag{6.18}$$

and from the general form (6.3) of variation, we conclude that (cf. (3.12) and (3.13))

$$\frac{\partial}{\partial t_r}\left(\hbar \frac{\psi'_\alpha(x)}{\psi_\alpha(x)}\right) = v_{d+r}(\overset{\alpha}{x}). \tag{6.19}$$

In turn, we have the following lemma.

Lemma 6.1

$$v_{d+r}(\overset{\alpha}{x}) = \frac{1}{2i\pi} \oint_{\mathcal{C}_D > x} B(\overset{\alpha}{x}, z) \frac{z^r}{r} dz, \quad r = 1, \ldots. \tag{6.20}$$

proof:

That the expression in (6.20) has the desired structure follows from the explicit form of the kernel B; we need only to verify the normalization conditions. Vanishing of $\mathcal{A}$-cycle integrals is obvious; we must verify

$$\delta_{d,l} = \frac{1}{2i\pi} \oint_{\mathcal{C}_D} x^{-l} v_{d+r}(x) dx = \frac{1}{(2i\pi)^2} \oint_{\mathcal{C}_D > x} dz \oint_{\mathcal{C}_D} dx \frac{z^r}{r} B(z, x) x^{-l}. \tag{6.21}$$

Interchanging the order of integration contours and using that $x^{-l}B(z,x) \sim x^{-l-2}$ as $x \to \infty$, we observe that the only nonzero contribution comes from the double pole at $x = z$, which gives

$$\frac{1}{2i\pi} \oint_{\mathcal{C}_D} dx\, x^{-l} \left(\frac{\partial}{\partial z} \frac{z^r}{r} \right)\bigg|_{z=x} = \frac{1}{2i\pi} \oint_{\mathcal{C}_D} dx x^{r-l-1} = \delta_{r,l}. \quad \square$$

$\square$

We now define the *operator of the loop insertion*

$$\frac{\partial}{\partial V(y)} := \sum_{r=1}^{\infty} r y^{-r-1} \frac{\partial}{\partial t_r}$$

applying which to $\hbar\psi'/\psi$, we obtain

$$\frac{\partial}{\partial V(y)} \left(\hbar \frac{\psi'_\alpha(x)}{\psi_\alpha(x)} \right) = \sum_{r=1}^{\infty} y^{-r-1} \oint_{y>\mathcal{C}_D} B(\overset{\alpha}{x}, z) z^r dz, \tag{6.22}$$

and since $\oint_{\mathcal{C}_D} B(\overset{\alpha}{x}, z) dz = 0$, we add the term with $r = 0$ into the sum obtaining

$$\oint_{y>\mathcal{C}_D} B(\overset{\alpha}{x}, z) \frac{1}{y-z} dz$$

in the right-hand side. Note that the point y lies between some infinity, say, ∞_β and the integration contour $\mathcal{C}_D$. Pulling the contour of integration through the point y to infinity we obtain zero due to the asymptotic conditions for $B(\overset{\alpha}{x}, \overset{\beta}{z})$, so the only nonvanishing contribution comes from the residue at $y = z$, which eventually gives

$$\frac{\partial}{\partial V(y)} \left(\hbar \frac{\psi'_\alpha(x)}{\psi_\alpha(x)} \right) = -\frac{1}{2} B(\overset{\alpha}{x}, \overset{\beta}{y}). \tag{6.23}$$

Correspondingly, since $\frac{\partial}{\partial V(y)} V'(x) = \frac{1}{(y-x)^2}$, we obtain for the *two-point correlation function* $W_2^{(0)}(x, y)$:

$$W_2^{(0)}(\overset{\alpha}{x}, \overset{\beta}{y}) := \frac{\partial}{\partial V(y)} \omega(\overset{\alpha}{x}) = -\frac{1}{2} B(\overset{\alpha}{x}, \overset{\beta}{y}) + \frac{1}{2(y-x)^2}. \tag{6.24}$$

6.3 Variation of higher correlation functions

Note that for all the above variations w.r.t. the flat coordinates, we have a cycle $\delta\omega^*$ and a (sector-independent) function $\Lambda^*_{\delta\omega}$ such that

$$\delta\omega(\overset{\alpha}{x}) = \int_{\delta\omega^*} B(\overset{\alpha}{x}, z)\, \Lambda^*_{\delta\omega}(z)\, dz. \tag{6.25}$$

The theorem below allows computing infinitesimal variations of any $W_n^{(h)}$ under a variation of the Schrödinger equation.

Theorem 6.1 *Under an infinitesimal deformation $U \to U + \delta U$, we have:*

$$\delta W_n^{(h)}(\overset{\alpha_1}{x}_1, \ldots, \overset{\alpha_n}{x}_n) = \int_{\delta\omega^*} W_{n+1}^{(h)}(\overset{\alpha_1}{x}_1, \ldots, \overset{\alpha_n}{x}_n, x')\, \Lambda^*(x')\, dx' \tag{6.26}$$

where $(\delta\omega^*, \Lambda^*_{\delta\omega})$ is the dual cycle to the deformation of the resolvent $\omega \to \omega + \delta\omega$.

proof:

We prove this theorem by induction. We begin with the loop equation for $W_n^{(h)}(\overset{\alpha}{x}, J)$:

$$\left(2\hbar\frac{\psi'_\alpha(x)}{\psi_\alpha(x)} + \hbar\partial_x\right) W_n^{(h)}(\overset{\alpha}{x}; J) + U_n^{(h)}(\overset{\alpha}{x}, \overset{\alpha}{x}; J) = P_n^{(h)}(x, J). \tag{6.27}$$

Taking a variation δ w.r.t. any of the flat coordinate, we have

$$\begin{aligned}&\left(2\hbar\frac{\psi'_\alpha(x)}{\psi_\alpha(x)} + \hbar\partial_x\right) \delta W_n^{(h)}(\overset{\alpha}{x}, J) + \left(2\delta\hbar\frac{\psi'_\alpha(x)}{\psi_\alpha(x)}\right) W_n^{(h)}(\overset{\alpha}{x}, J) + \delta U_n^{(h)}(\overset{\alpha}{x}, \overset{\alpha}{x}; J)\\ &= \delta P_n^{(h)}(x, J),\end{aligned} \tag{6.28}$$

where $\delta P_n^{(h)}(x, J)$ is a polynomial in x of degree at most $d-2$. Here both $\delta U_n^{(h)}(\overset{\alpha}{x}, \overset{\alpha}{x}; J)$ and $2\delta\hbar\frac{\psi'_\alpha(x)}{\psi_\alpha(x)} = \int_{\delta\omega^*} 2B(\overset{\alpha}{x}, x')\Lambda^*(x')dx'$ can be expressed by the induction assumption in the dual-cycle-integration form; meanwhile

$$\delta U_n^{(h)}(\overset{\alpha}{x}, \overset{\alpha}{x}; J) = \int_{\delta\omega^*} U_{n+1}^{(h)}(\overset{\alpha}{x}, \overset{\alpha}{x}; J, x')\Lambda^*(x')dx' - \int_{\delta\omega^*} 2B(\overset{\alpha}{x}, x')\Lambda^*(x')dx' \cdot W_n^{(h)}(\overset{\alpha}{x}, J), \tag{6.29}$$

because no term containing the two-point correlation function $W_2^{(0)}(\overset{\alpha}{x}, x')$ enters $\delta U_n^{(h)}(\overset{\alpha}{x}, \overset{\alpha}{x}; J)$.

Using the loop equation of form (6.27) relating $W_{n+1}^{(h)}(\overset{\alpha}{x}; J, x')$ and $U_{n+1}^{(h)}(\overset{\alpha}{x}, \overset{\alpha}{x}; J, x')$, we observe that the second term in the r.h.s. of (6.29) cancels the contribution of $2\delta\hbar\frac{\psi'_\alpha(x)}{\psi_\alpha(x)}$, and we obtain that

$$\begin{aligned}&\left(2\hbar\frac{\psi'_\alpha(x)}{\psi_\alpha(x)} + \hbar\partial_x\right)\left(\int_{\omega^*} W_{n+1}^{(h)}(\overset{\alpha}{x}, J, x')\Lambda^*(x')dx' - \delta W_n^{(h)}(\overset{\alpha}{x}, J)\right)\\ = \;& \delta P_n^{(h)}(x, J) - \int_{\omega^*} P_{n+1}^{(h)}(x, J, x')\Lambda^*(x')dx'\\ = \;& \sum_{i=1}^{d-1} \alpha_i(J)\, h_i(x)\end{aligned} \tag{6.30}$$

with the r.h.s. being a polynomial in x of degree not higher than $d-2$ expressed in the basis of the polynomials $h_i(x)$. Using 6.4, we have

$$\int_{\omega^*} W_{n+1}^{(h)}(\overset{\alpha}{x}, J, x')\Lambda^*(x')dx' - \delta W_n^{(h)}(\overset{\alpha}{x}, J) = \sum_{i=1}^{d-1} \alpha_i(J)\, v_i(x) \tag{6.31}$$

but since both $W_n^{(h)}(x,J)$ and $W_{n+1}^{(h)}(x,J,x')$ have vanishing $\mathcal{A}$-cycle integrals, we have that $\alpha_i = 0$, i.e.

$$\delta W_n^{(h)}(\overset{\alpha}{x},J) = \int_{\omega^*} W_{n+1}^{(h)}(\overset{\alpha}{x},J,x')\Lambda^*(x')dx' \tag{6.32}$$

□

Corollary 6.1

$$\frac{\partial W_n^{(h)}(J)}{\partial \epsilon_\alpha} = \oint_{\mathcal{B}_\alpha} W_{n+1}^{(h)}(J,x')dx' \quad \textit{for any} \quad n \geq 0, h \geq 0.$$

7 Classical and quantum geometry: summary

In the table below we summarize the comparison between items in classical algebraic geometry and their quantum counterparts.

	classical $\hbar = 0$	**quantum**
plane curve:	$E(x,y) = \sum_{i,j} E_{i,j} x^i\, y^j$ $E(x,y) = 0$	$E(x,y) = \sum_{i,j} E_{i,j} x^i\, y^j\,, \quad [y,x] = \hbar$ $E(x,\hbar\partial_x)\psi = 0$
hyperelliptical plane curve:	$y^2 = U(x)$ $\deg U = 2d$	$y^2 - U(x)\,, \quad [y,x] = \hbar,$ $\hbar^2\psi'' = U\,\psi$
Potential:	$V'(x) = 2(\sqrt{U(x)})_+$	
resolvent:	$\omega(x) = V'(x)/2 + y.$	$\omega(\overset{\alpha}{x}) = V'(x)/2 + \hbar\frac{\psi_\alpha'(x)}{\psi_\alpha(x)}.$
physical sheet(s):	$y \sim_\infty -\frac{1}{2}V'(x)$, $\omega \sim t_0/x$	$\hbar\psi'/\psi \sim_\infty -\frac{1}{2}V'(x)$, $\omega \sim t_0/x$ sectors where $\psi \sim e^{-\frac{V}{2\hbar}}$
branchpoints:	simple zeroes of $U(x)$ $U(a_i) = 0$, $U'(a_i) \neq 0$ $i = 1, \dots, 2d+2$	half-lines of accumulations of zeroes of ψ L_i, $i = 1, \dots, 2d+2$
double points:	double zeroes of $U(x)$ $U(\widehat{a}_i) = 0$, $U'(\widehat{a}_i) = 0$	half-lines without accumulations of zeroes of ψ (?)
genus $g = -1$	degenerate surface	$\psi\, \mathrm{e}^{V/2\hbar}$ =polynomial

	classical $\hbar = 0$	quantum
$\mathcal{A}_\alpha$-cycles $\alpha = 1, \dots, g$	surround pairs of branchpoints	surround pairs of half-lines of accumulating zeroes
extra $\mathcal{A}_d$-cycle $\alpha = d$	surrounds last pair of branchpoints	surrounds last pair of half-lines of accumulating zeroes
$\mathcal{B}$-cycles	$\mathcal{A}_i \cap \mathcal{B}_j = \delta_{i,j}$	
Holomorphic forms, 1st kind differentials	$v_i(x) = \frac{-h_i(x)}{2\sqrt{U(x)}}$ h_i =polynomials, $\deg h_i \leq d-2$ normalized: $\oint_{\mathcal{A}_\alpha} v_i(x)\, dx = \delta_{\alpha,i}$, $\alpha = 1, \dots, d-1$	$v_i(\overset{\alpha}{x}) = \frac{1}{\hbar\psi_\alpha^2(x)} \int_{\infty_\alpha}^x \psi_\alpha^2(x')\, h_i(x')\, dx'$
Period matrix	$\tau_{i,j} = \oint_{\mathcal{B}_j} v_i \quad , i,j = 1, \dots, \mathfrak{g}\ , \qquad \tau_{i,j} = \tau_{j,i}$	
Filling fractions	$2i\pi\, \epsilon_\alpha = \oint_{\mathcal{A}_\alpha} \omega(x) dx \quad , \alpha = 1, \dots, \mathfrak{g}, \qquad \epsilon_d = t_0 - \sum_{\alpha=1}^{\mathfrak{g}} \epsilon_\alpha$	
3rd kind form	$G(x,z) \sim_{x\to z} 1/(z-x)$ $G(\overset{\alpha}{x}, \overset{\beta}{z}) = (2\omega(\overset{\beta}{z}) - V'(z) - \hbar\partial_z) K(\overset{\alpha}{x}, z)$	
Recursion kernel	$K(\overset{\alpha}{x}, z)$ $K(\overset{\alpha}{x}, z) = \widehat{K}(\overset{\alpha}{x}, z) - \sum_i v_i(\overset{\alpha}{x})\, C_i(z)$ $C_i(z) = \oint_{\mathcal{A}_i} \widehat{K}(x', z) dx'$ $\widehat{K}(x,z) = \frac{1}{z-x} \frac{1}{2\sqrt{U(x)}}$	$\widehat{K}(\overset{\alpha}{x}, z) = \frac{1}{\hbar\psi_\alpha^2(x)} \int_{\infty_\alpha}^x \frac{\psi_\alpha^2(x')dx'}{x'-z}$
Bergman kernel 2nd kind differential	$B(\overset{\alpha}{x}, \overset{\beta}{z}) = -\frac{1}{2}\, \partial_z\, G(\overset{\alpha}{x}, \overset{\beta}{z})$ $B(\overset{\alpha}{x}, \overset{\alpha}{z}) \sim 1/2(x-z)^2$	
Symmetry:	$B(\overset{\alpha}{x}, \overset{\beta}{z}) = B(\overset{\beta}{z}, \overset{\alpha}{x})$	
	$\oint_{\mathcal{A}_i} B(x, \overset{\beta}{z}) dx = 0$ $\oint_{\mathcal{B}_i} B(x, \overset{\beta}{z}) dx = 2i\pi\, v_i(\overset{\beta}{z})$	
Meromorphic forms	$\mathcal{R}(x)dx = \frac{r(x)dx}{2\sqrt{U(x)}}$ $r(x)$ =rational with poles r_i, $r(x) = O(x^{d-2})$	$\mathcal{R}(\overset{\alpha}{x}) = \frac{1}{\hbar\psi_\alpha^2(x)} \int_{\infty_\alpha}^x r(x')\, \psi\alpha^2(x')\, dx'$ $\mathrm{Res}_{r_i}\, r(x')\psi^2(x') = 0$
Higher correlators	$W_{n+1}^{(h)}(\overset{\alpha}{x}, J) = \sum_i \frac{1}{2i\pi} \oint_{\mathcal{C}_i} K(\overset{\alpha}{x}, z)\, dz \Big(W_{n+2}^{(h-1)}(z,z,J)$ $+ \sum'_{s+s'=h,\, I\uplus I'=J} W_{1+\lvert I\rvert}^{(s)}(z,I) W_{1+\lvert I'\rvert}^{(s')}(z,I') \Big)$ where $\mathcal{C}_i$ surrounds the branchpoint L_i, $\quad \cup_i \frac{1}{2i\pi} \oint_{\mathcal{C}_i} = \oint_{\mathcal{C}_D}$	
Symmetry	$W_n^{(g)}(x_1, x_2, \dots, x_n) = W_n^{(g)}(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})\ , \qquad \sigma \in S_n$	
Variations and dual cycle	$U(x) \to U(x) + \delta U(x)$ δU^*: $\delta\omega(\overset{\alpha}{x}) = \int_{\delta U^*} B(\overset{\alpha}{x}, x')\, \Lambda_{\delta U}(x')\ dx'$	
$\delta V' = \sum \delta t_k\, x^{k-1}$	$\delta_{t_k}\omega(\overset{\alpha}{x}) = \oint_{\mathcal{C}_D} B(\overset{\alpha}{x}, x')\, \frac{x'^k}{k}\, dx'$	
variation δt_0	$\delta_{t_0}\omega(\overset{\alpha}{x}) = \int_{\infty_{\tilde{d}_-}}^{\infty_{\tilde{d}_+}} B(\overset{\alpha}{x}, x')\, dx'$	
variation $\delta\epsilon_i$	$\delta_{\epsilon_i}\omega(\overset{\alpha}{x}) = \oint_{\mathcal{B}_i} B(\overset{\alpha}{x}, x')\, dx'$	
Variations of higher correlators	$\delta W_n^{(h)}(x_1, \dots, x_n) = \int_{\delta U^*} W_{n+1}^{(h)}(x_1, \dots, x_n, x')\ \Lambda_{\delta U}(x')\ dx'$	

8 Application: Matrix models

The main reason for introducing $W_n^{(h)}$ is that they satisfy the loop equations for the random β-eigenvalue ensembles. We can therefore identify them with the correlation functions (resolvents) of these ensembles.

Consider a (possibly formal) matrix integral:

$$Z = \int_{E_{N,\beta}} dM \, \mathrm{e}^{-\frac{N\sqrt{\beta}}{t_0} \operatorname{tr} V(M)} \tag{8.1}$$

where $V(x)$ is some polynomial, and where $E_{N,1} = H_N$ is the set of hermitian matrices of size N, $E_{N,1/2}$ is the set of real symmetric matrices of size N and $E_{N,2}$ is the set of quaternion self dual matrices of size N (see [17]).

Alternatively, we can integrate over the angular part and get an integral over eigenvalues only [17]:

$$Z = \int d\lambda_1 \dots d\lambda_N \; |\Delta(\lambda)|^{2\beta} \; \prod_{i=1}^{N} \mathrm{e}^{-\frac{N\sqrt{\beta}}{t_0} V(\lambda_i)}, \tag{8.2}$$

where $\Delta(\lambda) = \prod_{i<j}(\lambda_j - \lambda_i)$ is the Vandermonde determinant.

We can then generalize the matrix model to arbitrary values of β taking the integral (8.2) as a definition of the β-model integral. For this, we take

$$\hbar = \frac{t_0}{N} \left(\sqrt{\beta} - \frac{1}{\sqrt{\beta}} \right). \tag{8.3}$$

Notice that $\hbar = 0$ correspond to the Hermitian case $\beta = 1$, and $\hbar \to -\hbar$ corresponds to $\beta \to 1/\beta$.

8.1 Correlation functions and loop equations

We define the correlation functions (the resolvents)

$$W_k(x_1, \dots, x_k) = \beta^{k/2} \; \left\langle \sum_{i_1,\dots,i_k} \frac{1}{x_1 - \lambda_{i_1}} \cdots \frac{1}{x_k - \lambda_{i_k}} \right\rangle_c \tag{8.4}$$

and

$$W_0 = \mathcal{F} = \log \mathcal{Z}. \tag{8.5}$$

When considering variations in the potential $V(x)$, we again assume these resolvents to satisfy the asymptotic conditions sectorwise, which means that they are defined also sectorwise.

And we assume (this is automatically true if we are considering formal matrix integrals), that there is a large N expansion of the type (where we assume $\hbar = O(1)$):

$$W_k(\overset{\alpha_1}{x}_1, \dots, \overset{\alpha_k}{x}_k) = \sum_{h=0}^{\infty} (N/t_0)^{2-2h-k} W_k^{(h)}(\overset{\alpha_1}{x}_1, \dots, \overset{\alpha_k}{x}_k) \tag{8.6}$$

$$W_0 = \mathcal{F} = \sum_{h=0}^{\infty} (N/t_0)^{2-2h} W_0^{(h)} \equiv \sum_{h=0}^{\infty} (N/t_0)^{2-2h} \mathcal{F}_h. \tag{8.7}$$

The loop equations are obtained by integration by parts, for example:

$$0 = \sum_i \int d\lambda_1 \dots d\lambda_N \frac{\partial}{\partial \lambda_i} \left(\frac{1}{x - \lambda_i} |\Delta(\lambda)|^{2\beta} \prod_j \mathrm{e}^{-\frac{N\sqrt{\beta}}{t_0} V(\lambda_j)} \right) \tag{8.8}$$

gives:

$$\begin{aligned}
0 &= \sum_i \left\langle \frac{1}{(x-\lambda_i)^2} + 2\beta \sum_{j\neq i} \frac{1}{x-\lambda_i} \frac{1}{\lambda_i - \lambda_j} - \frac{N\sqrt{\beta}}{t_0} \frac{V'(\lambda_i)}{x-\lambda_i} \right\rangle \\
&= \sum_i \left\langle \frac{1}{(x-\lambda_i)^2} + \beta \sum_{j\neq i} \frac{1}{x-\lambda_i} \frac{1}{x - \lambda_j} - \frac{N\sqrt{\beta}}{t_0} \frac{V'(\lambda_i)}{x-\lambda_i} \right\rangle \\
&= \sum_i \left\langle \frac{1-\beta}{(x-\lambda_i)^2} + \beta \sum_{j} \frac{1}{x-\lambda_i} \frac{1}{x - \lambda_j} - \frac{N\sqrt{\beta}}{t_0} \frac{V'(\lambda_i)}{x-\lambda_i} \right\rangle \\
&= (\beta-1)\frac{1}{\sqrt{\beta}} W_1'(x) + \beta\left(\frac{1}{\beta} W_1^2(x) + \frac{1}{\beta} W_2(x,x)\right) \\
&\qquad - \frac{N\sqrt{\beta}}{t_0} \left(\frac{1}{\sqrt{\beta}} V'(x) W_1(x) - \sum_i \left\langle \frac{V'(x) - V'(\lambda_i)}{x-\lambda_i} \right\rangle \right)
\end{aligned} \tag{8.9}$$

We define the polynomial

$$P_1(x) = \sqrt{\beta} \sum_i \left\langle \frac{V'(x) - V'(\lambda_i)}{x-\lambda_i} \right\rangle = (V'\, W_1)_+. \tag{8.10}$$

We thus have the loop equation of [3]

$$W_1^2(x) + \hbar \frac{N}{t_0} W_1'(x) + W_2(x,x) = \frac{N}{t_0} \left(V'(x) W_1(x) - P_1(x) \right) \tag{8.11}$$

Using the expansion (8.6) we come to the Ricatti equation

$$W_1^{(0)}(x)^2 + \hbar\, \partial_x W_1^{(0)}(x) = V'(x) W_1^{(0)}(x) - P_1^{(0)}(x) \tag{8.12}$$

satisfied by $\omega(x)$:

$$W_1^{(0)}(x) = \omega(x). \tag{8.13}$$

The correlation functions of β-eigenvalue models obey therefore the topological recursion of definition 5.1.

8.2 Variation w.r.t. $\hbar$

In this subsection, we use the analogy with the β-eigenvalue ensemble to hint the possible form of the last remaining building block of our construction, which is the variation w.r.t. $\hbar$, the exponent of the Vandermonde determinant in (8.2). Up to irrelevant multipliers, we can consider $\beta(\partial/\partial\beta)$ instead of $\hbar(\partial/\partial\hbar)$, for which we have

$$\beta\frac{\partial}{\partial\beta}\log\mathcal{Z}\sim\frac{2\beta}{\mathcal{Z}}\int d\lambda_1\dots d\lambda_N\ \Delta(\lambda)^{2\beta}\log|\Delta(\lambda)|\prod_{i=1}^{N}e^{-\frac{N\sqrt{\beta}}{t_0}V(\lambda_i)},\tag{8.14}$$

so the logarithm of the Vandermonde determinant appears.

It seems impossible to construct such a term from $W_1(\overset{\alpha}{x})$, but we can use the *two*-point correlation function $W_2(\overset{\alpha}{x},\overset{\gamma}{y})$ instead. Adopting a β-model inspired definition of $W_2(\overset{\alpha}{x},\overset{\beta}{y})$ as a two-resolvent correlation function (not necessarily connected),

$$W_2(x,y)=\frac{1}{\mathcal{Z}}\int D_N\lambda\sum_{i=1}^{N}\frac{1}{x-\lambda_i}\sum_{j=1}^{N}\frac{1}{y-\lambda_j}|\Delta(\lambda)|^{\hbar N}e^{-\frac{N\sqrt{\beta}}{t_0}V(\lambda)}$$

we then introduce the regularization (both IR and UV, if speaking in the physical terms). At this point, we also split all the eigenvalues λ_i into clusters each of which corresponds to some sector S_γ; for each term $1/(y-\lambda_i)$ we then integrate w.r.t. y from Λ_γ to $x+\delta_\gamma$ along the straight lines all of which are parallel; the regularization parameters depend only on the sector number γ, and the limit of removed regularization corresponds to $\Lambda_\gamma\to\infty_\gamma$ and $\delta_\gamma\to 0$. We then obtain

$$\begin{aligned}\frac{2\beta}{\mathcal{Z}}&\int_{\Lambda_\gamma}^{x+\delta_\gamma}d\xi W_2(\overset{\alpha}{x},\overset{\gamma}{\xi})\sim\\ &\sim\int d\lambda_1\dots d\lambda_N\ \Delta(\lambda)^{2\beta}\sum_{i=1}^{N}\frac{1}{x-\lambda_i}\sum_{\gamma}\sum_{j_\gamma=1}^{\epsilon_\gamma}\int_{\Lambda_\gamma}^{x+\delta_\gamma}\frac{d\xi}{\xi-\lambda_{j_\gamma}}\prod_{i=1}^{N}e^{-\frac{N\sqrt{\beta}}{t_0}V(\lambda_i)}\\ &=\int d\lambda_1\dots d\lambda_N\ \Delta(\lambda)^{2\beta}\sum_{i=1}^{N}\frac{1}{x-\lambda_i}\times\\ &\quad\times\sum_{\gamma}\sum_{j_\gamma=1}^{\epsilon_\gamma}\Big[\log\left|x+\delta_\gamma-\lambda_{j_\gamma}\right|-\log|\Lambda_\gamma|+O(1/\Lambda_\gamma)\Big]\prod_{i=1}^{N}e^{-\frac{N\sqrt{\beta}}{t_0}V(\lambda_i)}.\end{aligned}\tag{8.15}$$

We now want to perform the integration over the variable x to obtain the expression (8.14). Obviously, we need to choose the integration contour in a rather specific way: we want it to encircle all the poles in λ_i in the variable x leaving outside all the logarithmic cuts from ∞_γ to $\lambda_i+\delta_\gamma$ in the corresponding sector (see Fig. 8.2). Given such a contour, we can then perform the integration w.r.t. x by residues at the points λ_i (recall that in the eigenvalue model pattern we do not have boundaries between sectors inside the complex plane as yet; they appear because of the collective effect of taking into account the λ poles and due to sectorwise regularization chosen). Evaluating the integral over x in (8.15) by residues at λ_i, we obtain

$$
\begin{aligned}
&\frac{2\beta}{\mathcal{Z}}\int d\lambda_1\dots d\lambda_N\ \Delta(\lambda)^{2\beta}\Big[\sum_{i=1}^{N}\sum_{\gamma}\sum_{j_\gamma=1}^{\epsilon_\gamma}\log\left|\lambda_i+\delta_\gamma-\lambda_{j_\gamma}\right|-\\
&\qquad -N\sum_{\gamma}\epsilon_\gamma\log|\Lambda_\gamma|+O(1/\Lambda_\gamma)\Big]\prod_{i=1}^{N}e^{-\frac{N\sqrt{\beta}}{t_0}V(\lambda_i)}\\
=\ &\frac{2\beta}{\mathcal{Z}}\int d\lambda_1\dots d\lambda_N\ \Delta(\lambda)^{2\beta}\left|\log\prod_{i\neq j}(\lambda_i-\lambda_j+\delta_\gamma)\right|-\\
&\qquad -2\beta N\sum_{\gamma}\epsilon_\gamma\log|\Lambda_\gamma|+2\beta\sum_{\gamma}\epsilon_\gamma\log|\delta_\gamma|, \qquad (8.16)
\end{aligned}
$$

where the first term in the r.h.s. gives the desired integral (8.14) as $\delta_\gamma\to 0$, and the last two terms are divergent in the limit of regularization removed. These two terms depend however only on the occupation numbers therefore contributing to potential-independent part of $\mathcal{F}_0$ only and we can remove them by the proper normalization.

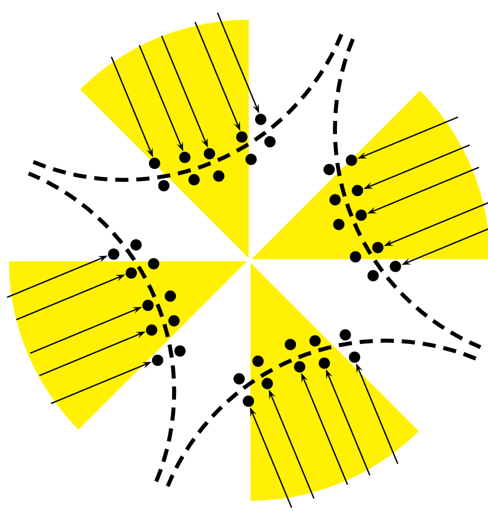


Figure 7: The origin of the integration contour $\mathcal{C}_D$ in the matrix-model concept. The inner dots are λ_i and the outer dots are $\lambda_i+\delta_\gamma$ ($\gamma=0,2,4,6$); thin arrowed lines are the logarithmic cuts.

9 The free energy

We use the variations and theorem 5.4 to define the $\mathcal{F}_h$.

9.1 The operator $\widehat{H}$

Theorem 5.4 gives:

$$(2-2h-n-\hbar\,\partial_\hbar)\,W_n^{(h)} = \left(t_0\,\partial_{t_0} + \sum_{k=1}^{d+1} t_k\,\partial_{t_k} + \sum_{i=1}^{g} \epsilon_i \partial_{\epsilon_i}\right) W_n^{(h)} \tag{9.1}$$

In section 6, we expressed the derivatives of $W_n^{(h)}$ as integrals of $W_{n+1}^{(h)}$ up to the action of $\hbar\frac{\partial}{\partial\hbar}$,

$$(2-2h-n-\hbar\,\partial_\hbar)\,W_n^{(h)} = \widehat{H}.\,W_{n+1}^{(h)} = \widehat{H}.\frac{\partial}{\partial V}\,W_n^{(h)} \tag{9.2}$$

where $\widehat{H}$ is the linear operator acting as follows:

$$\widehat{H}.f(x) = t_0 \oint_{\widetilde{\mathcal{B}}_d} f + \sum_{j=1}^{d+1} \int_{\mathcal{C}_D} \frac{t_j\,x^j}{j}\, f + \sum_{i=1}^{\mathfrak{g}} \epsilon_i \oint_{\mathcal{B}_i} f. \tag{9.3}$$

We define $W_0^{(h)} = \mathcal{F}_h$ for $n=0$ and $h \geq 2$ as

Definition 9.1 *The free energy $\mathcal{F}_h$ for $h \geq 2$ is the functions for which*

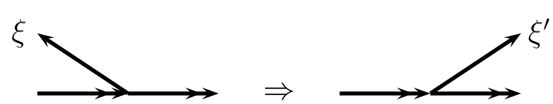
$$(2-2h-\hbar\,\partial_\hbar)\,\mathcal{F}_h = \widehat{H}.\,W_1^{(h)} \tag{9.4}$$

9.2 The derivative $\hbar\frac{\partial}{\partial\hbar}$

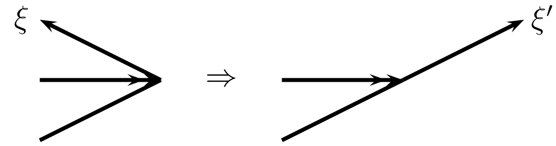
The matrix-model considerations in preceding section imply that constructing the derivative in $\hbar$ of the correlation function $W_n^{(h)}(J)$ would involve resolvents of order $n+2$. That is,

$$\begin{aligned}\hbar\frac{\partial}{\partial\hbar}W_n^{(h)}(J) \;\; = \;\; & \int_{\mathcal{C}_{D_\xi}} d\xi\,\Bigg[\int_\infty^\xi W_{n+2}^{(h-1)}(\overline{\xi}', \xi, J)d\xi' + \\ & + \sum_{r=0}^{h}\sum_{I\subseteq J}\int_\infty^\xi W_{|I|+1}^{(r)}(\overline{\xi}', I)d\xi' \cdot W_{n-|I|+1}^{(h-r)}(\xi, J/I)\Bigg],\end{aligned} \tag{9.5}$$

where $\overline{\xi}$ must be taken to be an "innermost" variable in the sense that taking into account that $\int^\xi B(\xi', y) = G(\xi, y)$, we replace all the appearances



and

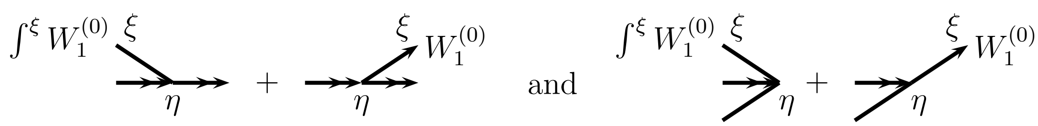


with *no additional factors*.

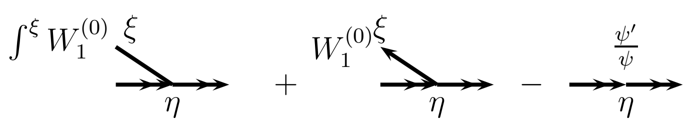
Note that the sum in (9.5) ranges all cases, not necessarily stable ones, so we begin with studying nonstable contributions to stable cases ($2h - 2 + n > 0$). Note that all these contributions then come from the second term in (9.5).

9.2.1 Case $r = 0$, $I = \emptyset$ and $r = h$, $I = J$

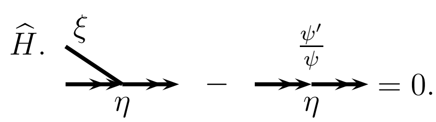
We consider the situation when $n \geq 1$. We can then fix x_1 to be the root of all the subtrees composed from the K-propagators and ξ can then be the variable of any of external B-legs. Then, the contribution in $W_n^{(h)}(J)$ comprises all the insertions



Let us consider the first case; the second one can be treated analogously. We push the integration over ξ through the one over η in the second diagram and as the result we obtain



Here the sum of the first two terms contains the integral of the total derivative of the function $\int^\xi W_1^{(0)}(\xi')d\xi' G(\eta,\xi)$, and since $G(\eta,\xi) \sim O(\xi^{-1})$ and $\int^\xi W_1^{(0)}(\xi')d\xi' \sim \int^\xi t_0 d\xi'/\xi' \sim t_0 \log|\xi|$, this contribution vanishes. Only the third contribution coming from the residue at $\xi = \eta$ survives, and this contribution is nothing but minus the action of the $\widehat{H}$ operator on the external leg $B(\eta,\xi)$, so



So, the total contribution of the two cases $r = 0$, $I = \emptyset$ and $r = h$, $I = J$ *exactly cancels the action of the* $\widehat{H}$ *operator.*

9.2.2 Case $r = 0$, $I = \{x_1\}$

We begin with the identity

$$\int_{x > \mathcal{C}_{D_\xi} > y} G(\overset{\alpha}{x}, \xi) K(\xi, y) = -K(\overset{\alpha}{x}, y). \tag{9.6}$$

(Here and hereafter inequalities of the type $x > \mathcal{C}_{D_\xi} > y$ indicates the mutual positions of points and integration contours.) Indeed, representing $G(\overset{\alpha}{x}, \overset{\gamma}{\xi}) = \psi_\gamma^2(\xi)\partial_\xi \frac{K(\overset{\alpha}{x},\xi)}{\psi_\gamma^2(\xi)}$ and integrating by parts, we obtain

$$\sum_\gamma K(\overset{\alpha}{x}, \xi) K(\overset{\gamma}{\xi}, y)\Big|_{\infty_{\tilde\gamma_-}}^{\infty_{\tilde\gamma_+}} - \int_{x > \mathcal{C}_{D_\xi} > y} K(\overset{\alpha}{x}, \xi) \left[\frac{1}{\xi - y} + \sum_j h_j(\xi) C_j(y)\right],$$

the substitution apparently gives zero, and in the second term only the residue at $\xi = y$ contributes thus producing (9.6). An obvious corollary is the second convolution formula

$$\int_{x > \mathcal{C}_{D_\xi} > y} G(\overset{\alpha}{x}, \xi) B(\xi, \overset{\beta}{y}) = -B(\overset{\alpha}{x}, \overset{\beta}{y}). \tag{9.7}$$

In the case $r = 0$, $I = \{x_1\}$, we have the diagram

$$x_1 \xrightarrow[\xi]{} \boxed{W_n^{(h)}(\xi, J/\{x_1\})} \quad = -W_n^{(h)}(J)$$

9.2.3 Case $r = h$, $I = J/\{x_n\}$

Here, we need another identity

$$\int_{x,y > \mathcal{C}_{D_\xi}} G(\overset{\alpha}{x}, \xi) B(\xi, \overset{\beta}{y}) = 0. \tag{9.8}$$

to obtain it, we represent the functions G and B through the kernel K, that is, we have

$$\begin{aligned}
&\int_{x,y > \mathcal{C}_{D_\xi}} G(\overset{\alpha}{x}, \xi) B(\xi, \overset{\beta}{y}) = \\
&= \int_{x,y > \mathcal{C}_{D_\xi}} \left(\partial_\xi - 2\frac{\psi'_\gamma(\xi)}{\psi_\gamma(\xi)}\right) K(\overset{\alpha}{x}, \xi) \partial_\xi \left(\partial_\xi - 2\frac{\psi'_\gamma(\xi)}{\psi_\gamma(\xi)}\right) K(\overset{\beta}{y}, \xi) \\
&= K(\overset{\alpha}{x}, \xi) B(\overset{\beta}{y}, \overset{\gamma}{\xi})\Big|_{\infty_-}^{\infty_+} - \int_{x,y > \mathcal{C}_{D_\xi}} K(\overset{\alpha}{x}, \xi) \left(\partial_\xi + 2\frac{\psi'_\gamma(\xi)}{\psi_\gamma(\xi)}\right) \partial_\xi \left(\partial_\xi - 2\frac{\psi'_\gamma(\xi)}{\psi_\gamma(\xi)}\right) K(\overset{\beta}{y}, \xi).
\end{aligned}$$

Here the substitution gives zero, and the third-order differential operator acting on the kernel $K(\overset{\beta}{y}, \xi)$ is again the Gelfand–Dikii operator (4.35) that does not depend on the sector γ. The integrand is also obviously regular at all zeros of ψ-functions, so the total integration over $\widetilde{\mathcal{A}}$-cycles just gives zero.

Therefore, the contribution of the case $r = h$, $I = J/\{x_n\}$ is zero, and the total contribution of all the unstable cases together with the action of the $\widehat{H}$-operator just gives minus one times the original contribution $W_n^{(h)}(J)$.

9.3 Examples of application of $\hbar\frac{\partial}{\partial\hbar}$

9.3.1 Reconstructing $W_n^{(0)}(J)$

We now apply formula (9.5) to reconstruct the correlation function $W_n^{(0)}(J)$. In the zero genus case, we must take into account only the nonconnected contributions (the second term in (9.5)) into account; we choose the root of the first term, $\int_\infty^\xi W^{(r)}_{|I|+1}(\overline{\xi}', I)d\xi'$, to be x_1, the point $\overline{\xi}$ is then the end of some other (nonrooted) leg $G(\eta, \overline{\xi})$ of the first diagram; for the second diagram we choose the root to be at the end ξ of leg with the corresponding propagator $K(\xi, \rho)$. As the result of the integration over ξ, we obtain using (9.6) that these two diagrams are sewed along the propagator $K(\eta, \rho)$ thus producing the *connected* diagram with the maximum subtree of propagators K rooted at the external point x_1. We may now ask the question *how many times* the given diagram can be obtained as a composition of two diagrams in the formula (9.5)? We obtain this diagram by first breaking it into two parts by cutting some of internal arrowed lines (including also the external line $K(x_1, \kappa)$ if we take into account the nonstable contributions) and then sewing again along the same line; obviously, we obtain this diagram as many times as the total number of arrowed lines (with the minus sign from (9.6)), which is $2 - n$ for $W_n^{(0)}(J)$. So, we see that adopting the definition (9.5) for the action of the operator $\hbar\frac{\partial}{\partial\hbar}$, we obtain

$$\left(\hbar\frac{\partial}{\partial\hbar} + \widehat{H}.\frac{\partial}{\partial V}\right) W_n^{(0)}(J) = (2-n)W_n^{(0)}(J), \tag{9.9}$$

which is a particular case of formula (9.2).

9.3.2 Acting on $\overline{W}_2^{(0)}(x_1, x_2)$

Here, we consider the action on a nonstable correlation function $\overline{W}_2^{(0)}(x_1, x_2) = B(\overset{\alpha_1}{x}_1, \overset{\alpha_2}{x}_2) - \frac{\delta_{\alpha_1,\alpha_2}}{(x_1-x_2)^2}$. Excluding the terms that compensate the action of $\widehat{H}$, we have that the action of $\hbar\frac{\partial}{\partial\hbar}$ gives

$$\begin{aligned}&\int_{x_1>\mathcal{C}_{D_\xi}>x_2} d\xi\, G(\overset{\alpha_1}{x}_1, \xi)\left(B(\xi, \overset{\alpha_2}{x}_2) - \frac{1}{(\xi-x_2)^2}\right) - \int_{x_1,x_2>\mathcal{C}_{D_\xi}} d\xi\,\frac{1}{x_1-\xi}B(\overset{\alpha_2}{x}_2, \xi) + x_1 \leftrightarrow x_2\\ &= -2B(\overset{\alpha_1}{x}_1, \overset{\alpha_2}{x}_2) + B(\overset{\alpha_1}{x}_1, \overset{\alpha_2}{x}_2) + B(\overset{\alpha_2}{x}_2, \overset{\alpha_1}{x}_1) = 0,\end{aligned} \tag{9.10}$$

which again is in accordance with formula (9.2).

9.3.3 Acting on $\mathcal{F}_1$

Here we demonstrate the first case of "reconstructing" the free energy term (although this is not true, to obtain the genuine term $\mathcal{F}_1$ we need other methods, which are still missing; we must however demonstrate that applying formula (9.2) we get zero perhaps up to some irrelevant regularizing factors). In this case, no nonstable terms contribute; the only contribution comes from the first term in (9.5), which gives

$$\left(\hbar\frac{\partial}{\partial\hbar} + \widehat{H}.\frac{\partial}{\partial V}\right)\mathcal{F}_1 = \int_{\mathcal{C}_{D_\xi}} d\xi\,\left(G(\xi,\bar{\xi}) - \frac{1}{\xi-\bar{\xi}}\right)$$
$$= \int_{\mathcal{C}_{D_\xi}} d\xi\,\left[\int_{\infty_\alpha}^{\xi+\delta_\alpha} d\xi'\,\frac{\partial}{\partial\xi}\frac{\psi_\alpha^2(\xi')/\psi_\alpha^2(\xi)-1}{\xi'-\xi} + \sum_j\int_{\infty_\alpha}^{\xi+\delta_\alpha} d\xi'\,h_j(\xi')\psi_\alpha^2(\xi')\left(\frac{C_j(\xi)}{\psi_\alpha^2(\xi)}\right)'.\right]$$

Integrating by parts in the second term, we obtain (up to terms of order $O(\delta_\alpha)$) $\int_{\mathcal{C}_{D_\xi}} d\xi\,h_j(\xi)C_j(\xi)$, and the integrand is sector-independent and nonsingular at zeros of ψ_α thus giving zero upon integration. In the first term, integrating by parts in the variable ξ' the term with $1/(\xi-\xi')^2)$ and taking into account that $\lim_{\xi'\to\xi}\frac{1}{\xi'-\xi}\left(\frac{\psi_\alpha^2(\xi')}{\psi_\alpha^2(\xi)}\right) = 2\frac{\psi'_\alpha(\xi)}{\psi_\alpha(\xi)}$, we obtain

$$\int_{\mathcal{C}_{D_\xi}} d\xi\,\left[-2\frac{\psi'_\alpha(\xi)}{\psi_\alpha(\xi)} + \int_{\infty_\alpha}^{x+\delta_\alpha} d\xi'\,\frac{2\frac{\psi'_\alpha(\xi')\psi_\alpha(\xi')}{\psi_\alpha^2(\xi)} - 2\frac{\psi'_\alpha(\xi)\psi_\alpha^2(\xi')}{\psi_\alpha^3(\xi)}}{\xi'-\xi}\right]$$
$$= \int_{\mathcal{C}_{D_\xi}} d\xi\,\left[-2\frac{\psi'_\alpha(\xi)}{\psi_\alpha(\xi)} + \int_{\infty_\alpha}^{x+\delta_\alpha} d\xi'\,\left(\frac{2}{\xi'-\xi}\frac{\psi'_\alpha(\xi')\psi_\alpha(\xi')}{\psi_\alpha^2(\xi)} + \frac{\psi_\alpha^2(\xi')}{\xi'-\xi}\left(\frac{1}{\psi_\alpha^2(\xi)}\right)'\right)\right]$$
$$= \int_{\mathcal{C}_{D_\xi}} d\xi\,\left[-2\frac{\psi'_\alpha(\xi)}{\psi_\alpha(\xi)} + \int_{\infty_\alpha}^{x+\delta_\alpha} d\xi'\,\left(\frac{2}{\xi'-\xi}\frac{\psi'_\alpha(\xi')\psi_\alpha(\xi')}{\psi_\alpha^2(\xi)} - \frac{\psi_\alpha^2(\xi')}{\psi_\alpha^2(\xi)}\frac{1}{(\xi'-\xi)^2}\right)\right]$$
$$= \int_{\mathcal{C}_{D_\xi}} d\xi\,\left[-2\frac{\psi'_\alpha(\xi)}{\psi_\alpha(\xi)} + \int_{\infty_\alpha}^{x+\delta_\alpha} \frac{2}{\xi'-\xi}\frac{\psi'_\alpha(\xi')\psi_\alpha(\xi')}{\psi_\alpha^2(\xi)}d\xi' + \frac{\psi_\alpha^2(\xi')}{\psi_\alpha^2(\xi)}d\frac{1}{\xi'-\xi}\right]$$
$$= \int_{\mathcal{C}_{D_\xi}} d\xi\,\left[-2\frac{\psi'_\alpha(\xi)}{\psi_\alpha(\xi)} + \int_{\infty_\alpha}^{x+\delta_\alpha} \frac{1}{\xi'-\xi}\frac{\partial}{\partial\xi'}\frac{\psi_\alpha^2(\xi')}{\psi_\alpha^2(\xi)}d\xi' + \frac{\psi_\alpha^2(\xi')}{\psi_\alpha^2(\xi)}d\frac{1}{\xi'-\xi}\right]$$
$$= \int_{\mathcal{C}_{D_\xi}} d\xi\,\left[-2\frac{\psi'_\alpha(\xi)}{\psi_\alpha(\xi)} + \frac{1}{\xi'-\xi}\frac{\psi_\alpha^2(\xi')}{\psi_\alpha^2(\xi)}\bigg|_{\infty_\alpha}^{x+\delta_\alpha}\right]$$
$$= \int_{\mathcal{C}_{D_\xi}} d\xi\,\left(\frac{1}{\delta_\alpha} + O(\delta_\alpha)\right),$$

and the result is a constant, which is divergent in the limit of regularization removed but is otherwise independent on all the variables (the same phenomenon occurs when calculating the corresponding action of the $\widehat{H}$ operator on $\mathcal{F}_1$ in the standard matrix models, see [4, 3]).

9.3.4 Acting on $W_1^{(1)}(x)$

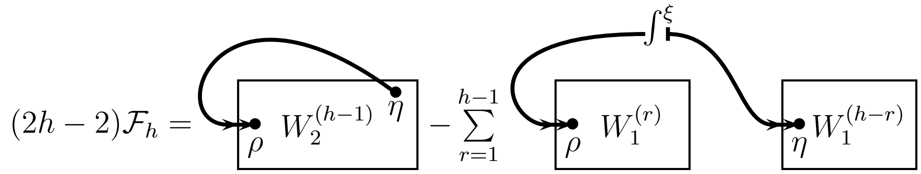
In this case, we have two possible contributions: the one from nonstable graphs gives $W_1^{(1)}(x)$ with the (desired) factor -1 whereas the second one would come from the first term in (9.5) originated from $W_3^{(0)}$ term, that is,

$$\int_{\mathcal{C}_{D_\xi}} d\xi \int_{\mathcal{C}_{D_\eta}} d\eta\, K(\overset{\alpha}{x},\eta)G(\eta,\overline{\xi})B(\eta,\xi), \tag{9.11}$$

where the contour of integration over η goes between the points $\overline{\xi}$ and ξ. We set integrals of this type to be zero, which provides the last required prescription for the diagrammatic technique describing the free energy terms $\mathcal{F}_h$.

9.4 The term $\mathcal{F}_h$

For the stable cases ($h \neq 0, 1$), we can now formulate the diagrammatic technique for the term $\mathcal{F}_h$. We need the diagrams describing the stable terms $W_1^{(r)}(\overline{\xi})$ and $W_1^{(h-r)}(\xi)$ with $1 \leq r \leq h-1$ and $W_2^{(h-1)}(\xi,\overline{\xi})$.



Here in the first term the sum ranges all the diagrams contributing to $W_2^{(h-1)}$ that have distinct vertices to which the external legs are attached; we then amputate both these legs and joint the vertices η and ρ (the ρ vertex is always the first three-valent vertex in the rooted tree) by the propagator $K(\eta,\rho)$. We took into account already the integration over ξ thus obtaining an extra minus sign. We cannot perform this integration that easily in the second term in which the integration over ξ is such that $\rho > \mathcal{C}_{D_\xi} > \eta$ and the symbol $\int^\xi$ indicates that we must insert the integration

$$\int_{\rho>\mathcal{C}_{D_\xi}>\eta} d\xi \int_{\infty_\alpha}^{\xi+\delta_\alpha} d\xi' K(\overset{\alpha}{\xi'},\rho)K(\overset{\alpha}{\xi},\eta)$$

between the integrations over the variables ρ and η.

10 Conclusion

We have defined quantum version of algebraic geometry notions, which allows us to solve the loop equations in the arbitrary β-ensemble case.

The notion of branchpoints become "blurred", a branchpoint is no longer a point, but an asymptotic accumulation line along which we integrate instead of taking the residue at the branch point.

Another surprising property pertains to the cohomology theory, which makes sense only if the cycle integral of any form depends only on the homology class of the cycle, i.e., we need all forms to have vanishing residues at the zeros s_i. This "no-monodromy" condition is automatically satisfied for our forms coming from the Schrödinger equation and it is equivalent to the set of Bethe ansatz equations satisfied by s_i, similar to what takes place in the Gaudin model [2].

In contrast to paper I, there is no explicit dependence on the chosen sector; however, even the total number of $\mathcal{A}$-cycles and rank of the period matrix may vary depending on the choice of cuts in the complex plane. This might pertain to that we do not have actual finite-genus (classical) Riemann surface: analytical continuation may never result in sewing the corresponding solutions to the Schrödinger equation, and we therefore deal with different (finite-genus) sections of an ambient infinite-genus surface. So, indeed, the genus is no longer deterministic.

Using the sectorwise approach, we can define the symplectic invariants; in Appendix D below we present the first nontrivial calculation of this sort: the dependence of the leading term on the occupation numbers.

In this paper, we restricted ourselves to the case of hyperelliptic curves, i.e. second order differential equations, that corresponds to 1-matrix model. The first straightforward generalization pertains to including the logarithmic potentials into consideration, which would produce the Nekrasov functions nonperturbatively in the parameter ϵ_2/ϵ_1. A more challenging problem is to generalize this approach to linear differential equations of any order, which would correspond to a 2-matrix β-ensemble model. In this case, we are also presumably able to define the notions of sheets, branchpoints, forms, and correlation functions. We expect also the preservation of the Bethe ansatz property ensuring a no-monodromy condition claiming that all cycle integrals depend only on the homology classes of cycles. The difference between the hyperelliptic case and the general case is comparable to the difference between the patterns of papers [9] and [6], i.e., the definition of the kernel K must be more involved and less explicit, but we postpone this discussion for further publications.

It would be also interesting to see whether the quantities $\mathcal{F}_h$ possess a symplectic invariance, or more precisely a "canonical invariance", i.e., whether they are invariant under any change $(x,y) \to (\widetilde{x},\widetilde{y})$ such that $[\widetilde{y},\widetilde{x}] = [y,x] = \hbar$.

Acknowledgments

We would like to thank O. Babelon, M. Bergère, G. Borrot, P. Di Francesco, V. Pasquier, A. Prats-Ferrer, A. Voros for useful and fruitful discussions on this subject. The work of L.Ch. is supported by the Russian Foundation for Basic Research grants 09-02-93105-CNRS_a and 09-01-12150-ofi_m, by the Grant for Supporting Leading Scientific Schools NSh-795.2008.1, and by the Program Mathematical Methods for Nonlinear Dynamics. The work of B.E. and O.M. is partly supported by the Enigma European network MRT-CT-2004-5652, by the ANR project Géométrie et intégrabilité en physique mathématique ANR-05-BLAN-0029-01, by the European Science Foundation through the Misgam program, by the Quebec government with the FQRNT.

A Appendix: Proof of theorem 5.2

We now prove theorem 5.2, that all $W_n^{(h)}$'s satisfy the loop equation, i.e.,

$$\begin{aligned} P_{n+1}^{(h)}(x; \overset{\alpha_1}{x}_1, \dots, \overset{\alpha_n}{x}_n) &= \hbar\left(2\frac{\psi'_\alpha(x)}{\psi_\alpha(x)} + \partial_x\right)\overline{W}_{n+1}^{(h)}(\overset{\alpha}{x}, \overset{\alpha_1}{x}_1, \dots, \overset{\alpha_n}{x}_n) \\ &+ \sum_{r=0}^{h}\sum_{I\subset J}{}' \overline{W}_{|I|+1}^{(r)}(\overset{\alpha}{x}, I)\overline{W}_{n-|I|+1}^{(h-r)}(\overset{\alpha}{x}, J/I) + \overline{W}_{n+2}^{(h-1)}(\overset{\alpha}{x}, \overset{\alpha}{x}, J) \\ &+ \sum_j \partial_{x_j}\left(\frac{\overline{W}_n^{(h)}(\overset{\alpha}{x}, J/\{x_j\})\delta_{\alpha,\alpha_j} - \overline{W}_n^{(h)}(\overset{\alpha_j}{x}_j, J/\{x_j\})}{(x-x_j)}\right) \end{aligned} \tag{A.1}$$

is a polynomial in x of degree at most $d-2$.

From the definition, we have (with U from (5.4))

$$W_{n+1}^{(g)}(\overset{\alpha}{x}, J) = \frac{1}{2i\pi}\oint_{\mathcal{C}} dz\, K(\overset{\alpha}{x}, z)\left(U_{n+2}^{(g-1)}(z, z, J) + \sum_j B(\overset{\alpha_j}{x}_j, z)W_n^{(g)}(z, J/\{x_j\})\right). \tag{A.2}$$

Acting by $\hbar\left(2\frac{\psi'_\alpha(x)}{\psi_\alpha(x)} + \partial_x\right)$ on $K(\overset{\alpha}{x}, z)$ gives $1/(x-z)+\sum_{j=1}^{g} h_j(x)C_j(z)$, and the second part is obviously polynomial satisfying assertions of the theorem. Pulling the contour of integration w.r.t. z to infinity (with x originally outside the integration contour) and taking into account that the integral at infinity vanishes thanks to the asymptotic conditions, we find that only the residue at $z = x$ and the residue at $z = x_j$ in the second term in the brackets give nonzero contributions; the result of integration reads

$$U_{n+2}^{(g-1)}(\overset{\alpha}{x}, \overset{\alpha}{x}, J) + \sum_j B(\overset{\alpha_j}{x}_j, \overset{\alpha}{x})W_n^{(g)}(\overset{\alpha}{x}, J/\{x_j\}) + \sum_j \frac{\partial}{\partial x_j}\frac{W_n^{(g)}(J)}{x-x_j},$$

so taking into account (5.3), we obtain the assertion of the theorem. $\square$

B Appendix: The symmetricity of $W_3^{(0)}$

Theorem B.1 *The three-point function $W_3^{(0)}$ is symmetric*

proof:

Introducing $Y := -2\hbar\psi'/\psi$, $W_3^{(0)}$ is by definition

$$\begin{aligned}
& W_3^{(0)}(\overset{\alpha_0}{x}_0, \overset{\alpha_1}{x}_1, \overset{\alpha_2}{x}_2) \\
= \quad & \frac{1}{i\pi}\oint_{\mathcal{C}_D} dx\, K(\overset{\alpha_0}{x}_0, x) B(\overset{\alpha_1}{x}_1, x) B(\overset{\alpha_2}{x}_2, x) \\
= \quad & \frac{1}{4i\pi}\oint_{\mathcal{C}_D} dx\, K_0\, G_1'\, G_2' \\
= \quad & \frac{1}{4i\pi}\oint_{\mathcal{C}_D} dx\, K_0\, ((\hbar K_1'' + Y K_1' + Y' K_1)(\hbar K_2'' + Y K_2' + Y' K_2)) \\
= \quad & \frac{1}{4i\pi}\oint_{\mathcal{C}_D} dx\, K_0\, (\, \hbar^2 K_1'' K_2'' + \hbar Y (K_1' K_2'' + K_1'' K_2') + \hbar Y' (K_1'' K_2 + K_2'' K_1) \\
& + Y^2 K_1' K_2' + Y Y' (K_1 K_2' + K_1' K_2) + Y'^2 K_1 K_2 \,)
\end{aligned} \tag{B.1}$$

where we have introduced a shorthand notation $K_p = K(x_p, x)$, $G_p = G(x_p, x)$, all the derivatives are w.r.t. x, and we omit indices indicating the sectors.

The combinations $K_0 K_1 K_2 f(x)$, where $f(x)$ is sector-independent ($f = 1, U, U', \ldots$), vanish upon integration w.r.t. x because each of K_i is also sector-independent w.r.t. x. We can then use the Ricatti equation $Y_i^2 = 2\hbar Y_i' + 4U$ to replace Y_i^2 by $2\hbar Y_i'$ and $Y_i Y_i'$ by $\hbar Y_i''$, which gives

$$\begin{aligned}
& W_3^{(0)}(x_0, x_1, x_2) \\
= \quad & \frac{1}{4i\pi}\oint_{\mathcal{C}_D} dx\, K_0 (\hbar Y (K_1' K_2'' + K_1'' K_2') + \hbar Y' (K_1'' K_2 + K_2'' K_1) \\
& + 2\hbar Y' K_1' K_2' + \hbar Y'' (K_1 K_2' + K_1' K_2) + Y'^2 K_1 K_2 \,) \\
= \quad & \frac{1}{4i\pi}\oint_{\mathcal{C}_D} dx\, K_0\, (\hbar Y (K_1' K_2')' + \hbar Y' (K_1 K_2)'' + \hbar Y'' (K_1 K_2)' + Y'^2 K_1 K_2 \,) \\
= \quad & \frac{1}{4i\pi}\oint_{\mathcal{C}_D} dx\, Y'^2 K_0 K_1 K_2 + \hbar (Y'' K_0 (K_1 K_2)' - (Y K_0)' K_1' K_2' - (Y' K_0)' (K_1 K_2)') \\
= \quad & \frac{1}{4i\pi}\oint_{\mathcal{C}_D} dx\, Y'^2 K_0 K_1 K_2 - \hbar ((Y K_0)' K_1' K_2' + Y' K_0' (K_1 K_2)') \\
= \quad & \frac{1}{4i\pi}\oint_{\mathcal{C}_D} dx\, Y'^2 K_0 K_1 K_2 - \hbar Y K_0' K_1' K_2' - \hbar Y' (K_0 K_1' K_2' + K_0' K_1 K_2' + K_0' K_1' K_2)
\end{aligned} \tag{B.2}$$

This expression is readily symmetric in x_0, x_1, x_2 as claimed in theorem 5.3. $\square$

C Appendix: Proof of theorem 5.3

Theorem 5.3 *Each $W_n^{(g)}$ is a symmetric function of all its arguments.*

proof:

The special case of $W_3^{(0)}$ was proved in appendix B above. The symmetricity of the two-point correlation function $\overline{W}_2^{(0)}$ was proved in Theorem 4.9.

For technical reason, it is easier to proceed with the proof for *nonconnected* correlation functions. We introduce two types of them:

- the correlation function

$$\widehat{W_n^{(h)}}(I) = \sum_{\substack{\text{partitions} \\ \{I_1,\dots,I_k\} \text{ of } I}} \prod_{j=1}^{k} W_{n_j}^{(h_j)}(I_j) \tag{C.1}$$

  that comprises partitions of only stable ($2h_j + n_j - 2 > 0$) type with $I_j \neq \emptyset$;

- the correlation function

$$\widetilde{W_n^{(h)}}(I) = \sum_{\substack{\text{partitions} \\ \{I_1,\dots,I_k\} \text{ of } I}} \prod_{j=1}^{k} W_{n_j}^{(h_j)}(I_j) \tag{C.2}$$

  that admits also two-point correlation functions $\overline{W}_2^{(0)}$ in the sums, $2h_j+n_j-2 \geq 0$, with $I_j \neq \emptyset$;

The symmetricity of all $\widehat{W_s^{(h')}}(I)$ with $s + 2h' \leq n + 2h$ obviously implies the symmetricity of $\widetilde{W_n^{(h)}}(I)$

It is obvious from the definition that $\widehat{W_{n+1}^{(h)}}(\overset{\alpha_0}{x_0}, \overset{\alpha_1}{x_1}, \dots, \overset{\alpha_n}{x_n})$ is symmetric in $x_1, x_2, \dots, x_n$, and therefore we need to show that (for $n \geq 1$):

$$\widehat{W_{n+1}^{(h)}}(\overset{\alpha_0}{x_0}, \overset{\alpha_1}{x_1}, J) - \widehat{W_{n+1}^{(h)}}(\overset{\alpha_1}{x_1}, \overset{\alpha_0}{x_0}, J) = 0, \tag{C.3}$$

where $J = \{\overset{\alpha_2}{x_2}, \dots, \overset{\alpha_n}{x_n}\}$.

The proof is by recursion on $-\chi = 2h - 2 + n$.

Assume that all $\widehat{W_k^{(h')}}$ and $\widetilde{W_k^{(h')}}$ with $2h' + k - 2 \leq 2h + n$ are symmetric. We have:

$$\begin{aligned} &\widehat{W_{n+1}^{(h)}}(\overset{\alpha_0}{x_0}, \overset{\alpha_1}{x_1}, J) \\ = \; &\frac{1}{2\pi i} \oint_{\mathcal{C}_{\mathcal{D}_x} > y} dx\, K(\overset{\alpha_0}{x_0}, x) \Big(\widetilde{W_{n+2}^{(h-1)}}(x, x, x_1, J) + 2B(\overset{\alpha_1}{x_1}, x) K(x, y) \widetilde{W_{n+1}^{(h-1)}}(y, J) \Big) \end{aligned} \tag{C.4}$$

We first consider the product of functions KBK in the second term: recalling that $B(\overset{\alpha_1}{x_1},\overset{\beta}{x}) = \partial_x(\partial_x - 2\frac{\psi'_\beta(x)}{\psi_\beta(x)})K(\overset{\alpha_1}{x_1},x)$ and integrating by parts, we obtain

$$
\begin{aligned}
& \frac{1}{2\pi i}\oint_{\mathcal{C}_{\mathcal{D}_x}>y} dx\, K(\overset{\alpha_0}{x_0},x)B(\overset{\alpha_1}{x_1},x)K(x,y) \\
= & -\frac{1}{2\pi i}\oint_{\mathcal{C}_{\mathcal{D}_x}>y} dx\, K'_x(\overset{\alpha_0}{x_0},x)K'_x(\overset{\alpha_1}{x_1},x)K(x,y) \\
& +\frac{1}{2\pi i}\oint_{\mathcal{C}_{\mathcal{D}_x}>y} dx\, K'_x(\overset{\alpha_0}{x_0},x)K(\overset{\alpha_1}{x_1},x)2\frac{\psi'(x)}{\psi(x)}K(x,y) \\
& -\frac{1}{2\pi i}\oint_{\mathcal{C}_{\mathcal{D}_x}>y} dx\, K(\overset{\alpha_0}{x_0},x)K'_x(\overset{\alpha_1}{x_1},x)K'_x(x,y) \\
& +\frac{1}{2\pi i}\oint_{\mathcal{C}_{\mathcal{D}_x}>y} dx\, K(\overset{\alpha_0}{x_0},x)K(\overset{\alpha_1}{x_1},x)2\frac{\psi'(x)}{\psi(x)}K'_x(x,y).
\end{aligned}
\tag{C.5}
$$

The first and the last terms in the right-hand side are already symmetric w.r.t. the replacement $x_0 \leftrightarrow x_1$ and we disregard them. Integrating by parts in the third term in the right-hand side we obtain one more symmetric term with $K(\overset{\alpha_0}{x_0},x)K(\overset{\alpha_1}{x_1},x)K''_{xx}(x,y)$ (which we can disregard as well) plus the term with $K'_x(\overset{\alpha_0}{x_0},x)K(\overset{\alpha_1}{x_1},x)K'_x(x,y)$. Combining the result with the second term, we obtain

$$
\begin{aligned}
& \frac{1}{2\pi i}\oint_{\mathcal{C}_{\mathcal{D}_x}>y} dx\, K'_x(\overset{\alpha_0}{x_0},x)K(\overset{\alpha_1}{x_1},x)\left(\partial_x + 2\frac{\psi'(x)}{\psi(x)}\right)K(x,y) \\
= & \frac{1}{2\pi i}\oint_{\mathcal{C}_{\mathcal{D}_x}>y} dx\, K'_x(\overset{\alpha_0}{x_0},x)K(\overset{\alpha_1}{x_1},x)\left(\frac{1}{x-y} + \sum_\beta h_\beta(x)C_\beta(y)\right),
\end{aligned}
\tag{C.6}
$$

where the integrand is sector-independent w.r.t. the variable x, so only the residue at $x = y$ (with the minus sign) contributes in the second term in (C.4), which therefore becomes

$$
-\frac{1}{2\pi i}\oint_{\mathcal{C}_{\mathcal{D}_y}} dy\, 2K'_y(\overset{\alpha_0}{x_0},y)K(\overset{\alpha_1}{x_1},y)\widetilde{W^{(h-1)}_{n+1}}(y,y,J).
\tag{C.7}
$$

For the first term in (C.4), we use the induction assumption writing it in the form

$$
\begin{aligned}
& \frac{1}{2\pi i}\oint_{\mathcal{C}_{\mathcal{D}_x}>y} dx\, \frac{1}{2\pi i}\oint_{\mathcal{C}_{\mathcal{D}_y}} dy\, K(\overset{\alpha_0}{x_0},x)K(\overset{\alpha_1}{x_1},y)\times \\
& \times\left(2B(x,y)^2\delta_{h=1,n=2} + 4B(x,y)\widetilde{W^{(h-1)}_{n+1}}(x,y,J)' + \widetilde{W^{(h-2)}_{n+3}}(x,x,y,y,J)'\right),
\end{aligned}
\tag{C.8}
$$

where the prime indicates that no propagators of $B(x,y)$ type enter the expression, and no singularity occurs in the corresponding terms upon interchanging the order of contour integration w.r.t. x and y. The last term is again obviously symmetric w.r.t. the replacement $x_0 \leftrightarrow x_1$.

The skew-symmetric part in the middle term is one-half of the residue coming from the double-pole $-1/(x-y)^2$ in the expression for $B(x,y)$ (it comes again with the minus sign due to the choice of contour ordering), so we obtain

$$\frac{1}{2\pi i}\oint_{\mathcal{C}_{\mathcal{D}_y}} dy\, 2K'_y(\overset{\alpha_0}{x_0}, y)K(\overset{\alpha_1}{x_1}, y)\widetilde{W^{(h-1)}_{n+1}}(y, y, J)' + \text{symmetric term}, \tag{C.9}$$

which exactly cancel the term in (C.7) except the only case $g = 1,\ n = 2$ in which we use that $B(x, y) = -1/(x - y)^2 + \overline{W}^{(0)}_2(x, y)$ as $x \to y$, so

$$2B(x, y)^2 = 2(x - y)^{-4} - 4(x - y)^{-2}\overline{W}^{(0)}_2(x, y) + \text{regular},$$

the most singular first term results in the integrand $K'''K$ that is sector-independent and vanishes, whereas the second term produces

$$\frac{1}{2\pi i}\oint_{\mathcal{C}_{\mathcal{D}_y}} dy\, 2K'_y(\overset{\alpha_0}{x_0}, y)K(\overset{\alpha_1}{x_1}, y)\overline{W}^{(0)}_2(y, y) + \text{symmetric term},$$

which kills the last remaining possible term in the expression (C.7). The theorem is proved. □

D Appendix: Calculating $\frac{\partial^3\mathcal{F}_0}{\partial t_0^3}$ in the Gaussian case

In this appendix, we calculate the singular part of the third derivative of $\mathcal{F}_0$ and integrate the answer to obtain the singular part of $\mathcal{F}_0$ itself. Although we explicitly calculate only the Gaussian model case, we propose the answer for the general model free energy singular part.

In the Gaussian model case with the potential $V(x) = x^2$, we have four sectors of solutions with the asymptotic directions $\pm\infty,\ \pm i\infty$. As the basic solutions we take $\psi_+(x)$ and $\psi_-(x)$ that decrease at the corresponding *imaginary* infinities $+i\infty$ and $-i\infty$. The real axis then plays the role of the $\widetilde{\mathcal{A}}$-cycle whereas the imaginary axis is the $\widetilde{\mathcal{B}}$-cycle.

We are interested in evaluating the singular part of the third-order derivative $\frac{\partial^3\mathcal{F}_0}{\partial t_0^3}$. Let us first find out about the origin of this singular behavior. Apparently, local singularities at finite t_0 appear when the solutions ψ_+ and ψ_- coincide, which happens when $\psi_\pm := \psi_n = H_n(ix)e^{x^2/2\hbar}$, where H_n are the Hermite polynomials, and

$$\hbar^2\partial_x^2\psi_n(x) = x^2\psi_n(x) + (2n + 1)\hbar\psi_n(x).$$

From Corollary 6.1 we have

$$\frac{\partial^3\mathcal{F}_0}{\partial t_0^3} = \frac{1}{(2\pi i)^3}\oint_{\mathcal{B}}\oint_{\mathcal{B}}\oint_{\mathcal{B}} dz_1\, dz_2\, dz_3\, W^{(0)}_3(z_1, z_2, z_3), \tag{D.1}$$

and using that $W_3^{(0)}(\overset{\alpha_1}{z_1},\overset{\alpha_2}{z_2},\overset{\alpha_3}{z_3}) = \oint_{\mathcal{C}_\mathcal{D}} d\xi K(\overset{\alpha_1}{z_1},\xi)B(\overset{\alpha_2}{z_2},\xi)B(\overset{\alpha_3}{z_3},\xi)$ and that no singularities appear when integrating over z_2 and z_3 we obtain that each integral gives, by Theorem 4.8, just $v_0(\overset{\alpha}{\xi})$ with $\alpha = \pm$ and

$$v_0(\overset{\pm}{\xi}) = C_0 \frac{1}{\psi_\pm^2(\xi)} \int_{\pm i\infty}^{\xi} \psi_\pm^2(\rho)d\rho \tag{D.2}$$

with the normalization constant C_0 such that

$$\int_{-\infty}^{+\infty} [v_0(\overset{-}{\xi}) - v_0(\overset{+}{\xi})] = 1.$$

Note that even in the case where $\psi_+ = \psi_-$ the functions $v_0(\overset{+}{\xi})$ and $v_0(\overset{-}{\xi})$ differ because of different lower limits of integrations, their difference is just $C_0 \frac{1}{\psi^2(\xi)} \int_{-i\infty}^{+i\infty} \psi^2(\rho)d\rho$, and the normalization constant C_0 at $\psi_+ = \psi_- = \psi_n$ is

$$C_0 = \left(\int_{-\infty}^{+\infty} \frac{1}{\psi_n^2(\xi)}d\xi\right)^{-1} \left(\int_{-i\infty}^{+i\infty} \psi_n^2(\rho)d\rho\right)^{-1}. \tag{D.3}$$

The remaining integral w.r.t. z_1 in (D.1) develops singularity when $\psi_\pm \to \psi_n$ because the function $K(\overset{\pm}{z_1},\xi)$ develops a logarithmic cut on the $\widetilde{\mathcal{B}}$-cycle, and using the explicit form (4.1) for the K-kernel (in this simplest case, $K = \widehat{K}$), we obtain

$$\frac{\partial^3 \mathcal{F}_0}{\partial t_0^3} = \sum_\pm \int_{\mathcal{C}_\xi^\pm} d\xi \frac{1}{2\pi i} \int_{\pm i\infty}^{\mp i\infty} dz \frac{1}{\hbar} \frac{1}{\psi_\pm^2(z)} \int_{\pm i\infty}^{z} d\rho \frac{\psi_\pm^2(\rho)}{\rho - \xi} v_0^2(\overset{\pm}{\xi}), \tag{D.4}$$

where the contour $\mathcal{C}_\xi^\pm$ goes between $\pm\infty$ and $\mp infty$ encircling the point ρ. The singularity occurs when ρ (and, correspondingly, z) tends to $-i\infty$ for ψ_+ and to $+i\infty$ for ψ_-; this singular part comes from the residue at $\xi = \rho$, and we obtain that the expression in (D.4) is

$$\sum_\pm \int_0^{\mp i\infty} dz \frac{1}{\hbar} \frac{1}{\psi_\pm^2(z)} \int_0^z d\rho \psi_\pm^2(\rho) v_0^2(\overset{\pm}{\rho}) + \text{regular}$$

and using the explicit expressions (D.2) and (D.3) for v_0, we obtain

$$\begin{aligned} &\text{sing.}\left(\frac{\partial^3 \mathcal{F}_0}{\partial t_0^3}\right) \\ &\quad = \sum_\pm \int_0^{\mp i\infty} dz \frac{1}{\hbar} \frac{1}{\psi_\pm^2(z)} \int_0^z d\rho \frac{1}{\psi_\pm^2(\rho)} \left[\int_{\pm i\infty}^{\rho} \psi_\pm^2(s)ds\right]^2 C_0^2, \end{aligned} \tag{D.5}$$

where the singularity occurs at the upper integration limit for z and ρ when $\psi_+, \psi_- \to \psi_n$, and the term in the square brackets is nonsingular in this limit, so we can replace it by its limiting value, which cancel exactly the corresponding term in the normalization constant C_0 (see (D.3); the integral w.r.t. z and ρ can be separated, and we obtain the final expression

$$\text{sing.}\left(\frac{\partial^3\mathcal{F}_0}{\partial t_0^3}\right)=\sum_{\pm}\frac{1}{2\hbar}\left[\int_0^{\mp i\infty}dz\frac{1}{\psi_\pm^2(z)}\right]^2\left[\int_{-\infty}^{+\infty}\frac{1}{\psi_\pm^2(x)}dx\right]^{-2}. \tag{D.6}$$

We now calculate t_0 when $\psi_+,\psi_-\to\psi_n$. Choosing $\psi_-(x)=\psi_+(x)\int_{-\infty}^{x}\frac{d\xi}{\psi_+^2(\xi)}$ and taking into account that the number of poles of solutions outside the $\widetilde{\mathcal{A}}$-cycle is n, we have

$$\begin{aligned} t_0 &= -\hbar n+\hbar\int_{-\infty}^{+\infty}\left(\frac{\psi_-'}{\psi_-}-\frac{\psi_+'}{\psi_+}\right)=-\hbar n+\hbar\int_{-\infty}^{+\infty}\frac{1}{\psi_+\psi_-}\\ &= -\hbar n+\hbar\int_{-\infty}^{+\infty}\frac{dz}{\psi_+^2(z)}\frac{1}{\int_{-i\infty}^{0}\frac{d\xi}{\psi_+^2(\xi)}+\int_0^z\frac{d\xi}{\psi_+^2(\xi)}}. \end{aligned}\tag{D.7}$$

The first integral in the denominator diverges as $\psi_+\to\psi_n$ and denoting this integral as Λ, we have

$$t_0\Big|_{\psi_+\to\psi_n}=-\hbar n+\hbar\frac{\int_{-\infty}^{+\infty}\frac{dz}{\psi_+^2(z)}}{\int_{-i\infty}^{0}\frac{d\xi}{\psi_+^2(\xi)}}+O(\Lambda^{-2}). \tag{D.8}$$

Comparing this expression with (D.6), we obtain

$$\text{sing.}\left(\frac{\partial^3\mathcal{F}_0}{\partial t_0^3}\right)=\frac{1}{\hbar(n+t_0/\hbar)^2},\quad n\in\mathbb{Z}_{+,0}, \tag{D.9}$$

that is, this derivative has double poles with the coefficient $1/\hbar$ at all points $t_0=-\hbar n$, $n=0,1,\ldots$. A function that exhibits such a behavior is obviously Γ-function, so we have that, up to an entire function,

$$\text{sing.}\left(\frac{\partial^3\mathcal{F}_0}{\partial t_0^3}\right)\simeq\frac{1}{\hbar}[\log\Gamma]''(t_0/\hbar), \tag{D.10}$$

and, in turn,

$$\text{sing.}\mathcal{F}_0\simeq\hbar^2[\textstyle\int\log\Gamma](t_0/\hbar). \tag{D.11}$$

Turning to the asymptotic behavior of $\int\log\Gamma(x)$ at large positive x we observe that the leading term is $\frac{1}{2}x^2\log x$, which is exactly what we might expect from the matrix-model-like arguments: we must be able to apply semiclassical approximation at large positive $t_0/\hbar$, and in this regime we have the leading asymptotic behavior of Gaussian matrix model, i.e., sing.$\mathcal{F}_0\simeq\frac{1}{2}t_0^2\log(t_0)$ modulo polynomial terms (of order not higher than two).

We may therefore put forward the following conjecture.

Conjecture D.1 *The singular part of $\mathcal{F}_0$ for any potential $V_{d+1}(x)$ has the form $\hbar^2\sum_{i=1}^{d}\frac{1}{2}[\int\log\Gamma](\widetilde{\epsilon}_i/\hbar)$ where $\widetilde{\epsilon}_i$ are the occupation numbers on the cycles $\widetilde{\mathcal{A}}_i$.*

References

[1] L. F. Alday, D. Gaiotto, and Y. Tachikawa, Liouville correlation function from four-dimensional gauge theories,
arXiv:0906.3219 [hep-th].

[2] O. Babelon, D. Talalaev, On the Bethe Ansatz for the Jaynes-Cummings-Gaudin model, hep-th/0703124.

[3] L. Chekhov, B. Eynard, Matrix eigenvalue model: Feynman graph technique for all genera, JHEP 0612 (2006) 026, math-ph/0604014.

[4] L. Chekhov, B. Eynard, Hermitean matrix model free energy: Feynman graph technique for all genera, JHEP 009P 0206/5, hep-th/0504116.

[5] L. O. Chekhov, B. Eynard, and O. Marchal, Topological expansion of the Bethe ansatz, and quantum algebraic geometry, arXiv:0911.1664.

[6] L. Chekhov, B. Eynard, N. Orantin, Free energy topological expansion for the 2-matrix model, JHEP 0612 (2006) 053, math-ph/0603003.

[7] I. Dumitriu, A. Edelman, Matrix models for beta ensembles, J. Math. Phys., **43** (2002) 5830–5847.

[8] T. Eguchi and K. Maruyoshi, Penner-type matrix model and Seiberg–Witten theory// arXiv:0911.4797v3[hep-th].

[9] B. Eynard, Topological expansion for the 1-hermitian matrix model correlation functions, JHEP/024A/0904, hep-th/0407261.

[10] B. Eynard, Asymptotics of skew orthogonal polynomials, J. Phys A. 34 (2001) 7591, cond-mat/0012046.

[11] B. Eynard, O. Marchal, Topological expansion of the Bethe ansatz, and non-commutative algebraic geometry, arXiv:0809.3367, JHEP03(2009)094

[12] B. Eynard, N. Orantin, Invariants of algebraic curves and topological expansion, Communications in Number Theory and Physics, Vol 1, Number 2, p347-452, math-ph/0702045.

[13] B. Eynard, A. Prats Ferrer, Topological expansion of the chain of matrices, math-ph: arxiv.0805.1368.

[14] I. M. Gel'fand and L. A. Dikii, Integrable nonlinear equations and the Liouville theorem, Funct. Anal. Appl. **13(1)** (1979) 6–15;

[15] I. Krichever, The tau-function of the universal Witham hierarchy, matrix models and topological field theories, Commun. Pure Appl. Math. **47** (1992) 437; hep-th/9205110.

[16] A. Mironov and A. Morozov, The power of Nekrasov functions, Phys. Lett. B**680**, 188 (2009) [arXiv:0908.2190[hep-th]].
A. Marshakov, A. Mironov and A. Morozov, On non-conformal limit of the AGT relations// arXiv:0909.2052[hep-th].
A. Marshakov, A. Mironov and A. Morozov, Zamolodchikov asymptotic formula and instanton expansion in $N = 2$ SUSY $N_f = 2N_c$ QCD // arXiv:0909.3338[hep-th].

[17] M. L. Mehta, Random matrices (3e edition), Pure and Applied Mathematics Series 142, Elsevier (London - 2004), 688 pp. ISBN 0120884097.

[18] N. A. Nekrasov, Selberg–Witten prepotential from instanton counting, Adv. Theor. Math. Phys. **7**, 831 (2004) [arXiv:hep-th/0206161].